# Bounds on the Norm of Wigner-type Random Matrices

László Erdős[1] and Peter Mühlbacher[1]

[1]IST Austria

February 13, 2018

Supported by ERC Advanced Grant No. 338804

# Central Question

What can be said about the statistical properties of the eigenvalues of a large random matrix? (Wigner)

$$H = \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1N} \\ H_{21} & H_{22} & \dots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \dots & H_{NN} \end{bmatrix} \Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_N) \text{ eigenvalues}$$

$N =$ size of the matrix, will go to infinity.

Analogy: Central limit theorem. $\frac{X_1 + \cdots + X_N}{\sqrt{N}} \sim \mathcal{N}(0, \sigma^2)$

# Motivation

- Physics
  - nuclear physics (Wigner)
  - mean field hopping mechanism with random transition rates
- Statistics
  - Wishart matrices
  - sample covariance matrices
  - neural networks (why are local minima good enough?)
- ODEs with random coefficients

# Wigner type matrices

$H$ real-symmetric or complex hermitian $N \times N$ matrix
entries are centered and independent (up to $H_{ij} = \overline{H_{ji}}$ for $i \leq j$)
with variance matrix

$$S_{ij} := \mathbb{E}|H_{ij}|^2$$

such that $S_{ij} \leq \frac{C}{N}$ uniformly in $i, j$ for $C > 0$ indep. of $N$.

The eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ are of order one (on average):

$$\mathbb{E}\frac{1}{N}\sum_i \lambda_i^2 = \mathbb{E}\frac{1}{N} TrH^2 = \frac{1}{N}\sum_{ij}\mathbb{E}|H_{ij}|^2 = O(1)$$

# Density of States (DoS)

Introduce the random measure, referred to as empirical density:

$$\rho_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

Analogously to the law of large numbers we have

$$\rho_N \approx \mathbb{E}\rho_N, \qquad \text{for } N \gg 1$$

Under suitable higher moment assumptions we have that

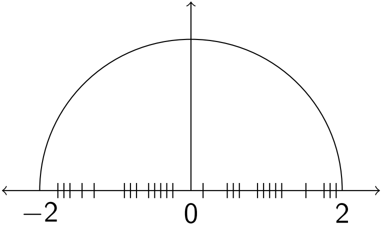
$$\mathbb{E}\rho_N \approx \widetilde{\rho}_N = \widetilde{\rho}_N(S) \qquad \text{for } N \gg 1,$$

where $\widetilde{\rho}_N := \lim_{\eta \downarrow 0} \frac{1}{\pi} \mathrm{Im} \langle \mathbf{m}(E + i\eta) \rangle$ is the inverse Stieltjes transform of $\mathbf{m}(z) = (m_1(z), \ldots, m_N(z))$, the unique (in $\mathbb{H}^N$) and deterministic solution to

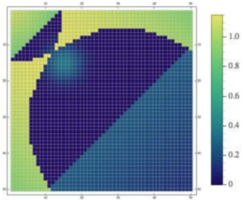
$$-\frac{1}{m_x} = z + (Sm)_x, \quad x = 1, \ldots, N \qquad \text{[QVE]}.$$

# Variance profile and density of states (DoS)

$\sum_i S_{ij} = 1 \quad \Leftrightarrow$




$\sum_i S_{ij} \neq 1$




$\Leftrightarrow$

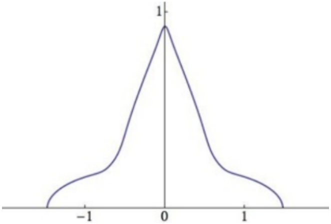



Figure: General variance profile $S_{ij} \Rightarrow$ not the semicircle anymore.

# Main theorem

$S$ symmetric $N \times N$ matrix of variances, $J \in \mathbb{N} \cup \{\infty\}$, $z_j := \frac{\|S^j\|}{\|S\|^j} \leq 1$, where $\|S\| := \max_i \sum_j S_{ij}$, and $w_c(J)$ is the smallest positive root of the function

$$\phi_J(w) := 1 - \frac{w}{2}\Big(1 + \sum_{j=1}^{J} \Big(\frac{w}{2}\Big)^j z_j + \sum_{j>J} \Big(\frac{w}{2}\Big)^j \Big).$$

Then we have

$$\max \operatorname{supp} \widetilde{\rho}_N \leq 2\frac{\|S\|^{\frac{1}{2}}}{w_c(J)}.$$

**Comparison to existing literature:**

- [Ajanki-Erdős-Krüger 2015]: $\max \operatorname{supp} \widetilde{\rho}_N \leq 2\|S\|^{\frac{1}{2}}$.
- [Ottolino 2017]: exact expression for $\max \operatorname{supp} \widetilde{\rho}_N$ in terms of a variational formula of $S$; our approach is different since it gives an explicit bound only in terms of powers of $S$.

# Corollary

This follows using local laws from [Erdős, Krüger, Schröder, 2017]. Let $H = H^{(N)}$ be a sequence of Wigner type matrices with variance matrices $S = S^{(N)}$, set

$$z_j := \limsup_{N\to\infty} \frac{\|[S^{(N)}]^j\|}{\|S^{(N)}\|^j} \leq 1$$

and for any $J \in \mathbb{N} \cup \{\infty\}$ let $w_c(J)$ the smallest positive root of $\phi_J$. Under finite moment assumption:

$$\forall q \in \mathbb{N} : \exists C_q : \mathbb{E} \max_{i,j,N} (\sqrt{N}|H_{ij}|)^q \leq C_q,$$

for any $\epsilon > 0$ (small) and any $D > 0$ (large) we have the following bound on the largest eigenvalue of $H$:

$$\mathbb{P}\left(|\lambda_{max}(H^{(N)})| \geq 2\frac{\|S^{(N)}\|^{\frac{1}{2}}}{w_c(J)} + \epsilon\right) \leq C(\epsilon, D)N^{-D}$$

for $C(\epsilon, D) = C(\epsilon, D, C^*, (C_q)_q)$ with $C^*$ such that $S_{ij} \leq \frac{C^*}{N}$.

# Numerics

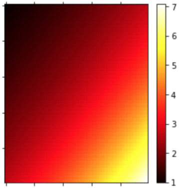



Figure: Variance profile $S_{ij} = \frac{1}{N} e^{\frac{i+j}{N}}$ for $N = 500$.

- The already known bound yields $2\|S\|^{1/2} \approx 4.316$.
- For $J = 50$ our bound gives $2\|S\|^{\frac{1}{2}}/w_c \approx 3.870$, an improvement by a factor of $w_c \approx 1.115$.
- The empirical average (number of samples $= 10$) of the largest eigenvalue (in absolute value) is $\approx 3.677$ (with empirical standard deviation of $\approx 0.047$)

# How to get the DoS in terms of $S$ - Stieltjes transform

**Definition** Let $\mu$ be a probability measure on $\mathbb{R}$. Its Stieltjes transform $m_\mu : \mathbb{H} \to \mathbb{C}$ is given by

$$m_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}$$

**Properties**

- $z \mapsto m_\mu(z)$ is analytic in $\mathbb{H}$ with image in $\mathbb{H}$:

$$\operatorname{Im} m_\mu(E + i\eta) = \eta \int_{\mathbb{R}} \frac{d\mu(x)}{(x - E)^2 + \eta^2} = (\pi\mu * P_\eta)(E),$$

  where $P_\eta(x) := \frac{1}{\pi}\frac{\eta}{x^2+\eta^2}$ is the Poisson kernel, satisfying
  - $\int_{\mathbb{R}} P_\eta(x)dx = 1$
  - $P_\eta(E) \rightharpoonup \delta_E$ as $\eta \downarrow 0$
- For every analytic $m : \mathbb{H} \to \mathbb{H}$ with $i\eta m(i\eta) \to -1$ as $\eta \to \infty$ and $|m(z)| \leq \frac{1}{\operatorname{Im}(z)}$ there is a measure $\mu$ such that

$$\frac{1}{\pi}\operatorname{Im} m(\cdot + i\eta) \rightharpoonup \mu \qquad \text{as } \eta \downarrow 0$$

# How to get the DoS in terms of $S$ - resolvents

**Obvious:** Trace of the resolvent $G$ of a hermitian matrix $H$ is the Stieltjes transform of its empirical spectral density.

$$\rho_N \overset{(\text{def})}{=} \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}, \qquad m_{\rho_N}(z) \overset{(\text{def})}{=} \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z} \overset{(\text{def})}{=} \frac{1}{N} TrG(z)$$

While, in general,

$$\lim_{\eta\downarrow 0} \operatorname{Im} m_{\rho_N}(E + i\eta)$$

does not exist, its expectation may exist

$$\rho(E) := \lim_{\eta\downarrow 0} \mathbb{E}\operatorname{Im} m_{\rho_N}(E + i\eta)$$

and gives the density of states (DoS) $\rho$.

# How to get the DoS in terms of $S$ - resolvents

**Highly non-trivial:** For $\mathbf{m}(z) = (m_1(z), \ldots, m_N(z))$ the unique and deterministic solution in $\mathbb{H}^N$ to

$$-\frac{1}{m_x} = z + (Sm)_x, \quad x = 1, \ldots, N \qquad \text{[QVE]}$$

it is shown in [Ajanki-Erdős-Krüger 2015] that

$$G(z) \overset{\text{(def)}}{=} \frac{1}{H - z}, \qquad G_{xx}(z) \approx m_x(z), \qquad \text{as } N \to \infty$$

So to compute the DoS for $H$, we need to solve the QVE for every $z \in \mathbb{H}$ and compute the inverse Stieltjes transform of

$$\langle \mathbf{m}(z) \rangle := \frac{1}{N} \sum_x m_x(z).$$

# Solving the QVE

**Case** $S_{ij} = \frac{1}{N}$:

$$-\frac{1}{m_x} \overset{\text{(def)}}{=} z + (Sm)_x = z + \langle \mathbf{m} \rangle, \quad \forall i$$

thus $m_x = \langle \mathbf{m} \rangle$ and $-\frac{1}{\langle \mathbf{m} \rangle} = z + \langle \mathbf{m} \rangle$ which gives the semicircle law.

**Case** $S_{ij} \neq \frac{1}{N}$: Note that [QVE] is equivalent to

$$-zm_x =: u_x = \frac{1}{1 - \frac{1}{z^2}(Su)_x} = \sum_{n=0}^{\infty} z^{-2n} \left( \sum_{y=1}^{N} S_{xy} u_y \right)^n \qquad (*)$$

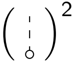
Immediate by recursively plugging in $(*)$:

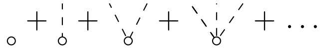
$$m_x = -\frac{1}{z} \sum_{n=0}^{\infty} c_{n,x} z^{-2n}, \quad c_{n,x} \in \mathbb{R}$$

$\Rightarrow -\overline{m_x(z)} = m_x(-\bar{z}) \Rightarrow \operatorname{Im} \langle \mathbf{m}(E + i\eta) \rangle = \operatorname{Im} \langle \mathbf{m}(-E + i\eta) \rangle$

$\Rightarrow$ DoS symmetric

# Solving the QVE

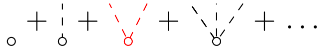
For  $:= \frac{1}{z^2}\sum_{y=1}^{N} S_{xy}\, u_y$,  $:=$  $, \dots,$ we have



$$\Leftrightarrow \quad \sum_{n(x)=0}^{\infty} z^{-2n(x)} \left( \sum_{y} S_{xy}\, u_y \right)^{n(x)} \stackrel{\text{(def)}}{=} u_x$$

# Solving the QVE

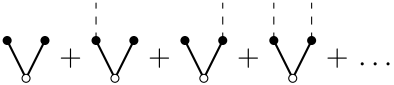
For  $:= \frac{1}{z^2}\sum_{y=1}^{N} S_{xy}\, u_y$,  $:= \Big($  $\Big)^2, \ldots,$ we have



$$\Leftrightarrow \quad \sum_{n(x)=0}^{\infty} z^{-2n(x)} \left( \sum_y S_{xy}\, u_y \right)^{n(x)} \overset{\text{(def)}}{=} u_x$$

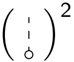
(choose $n(x) = 2$)

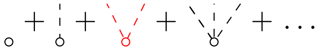
$$\Downarrow$$



$$\Leftrightarrow \quad z^{-4} \underbrace{\left( \sum_a S_{xa}\, u_a \right)^2}_{\sum_{a_1 a_2} S_{xa_1} S_{xa_2}\, u_{a_1} u_{a_2}}$$

# Solving the QVE

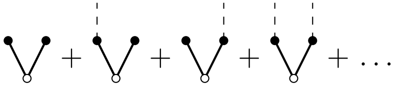
For  $:= \frac{1}{z^2}\sum_{y=1}^{N} S_{xy}\, u_y$,  $:=$ , ..., we have



$$\Leftrightarrow \quad \sum_{n(x)=0}^{\infty} z^{-2n(x)} \left( \sum_{y} S_{xy}\, u_y \right)^{n(x)} \overset{\text{(def)}}{=} u_x$$

$$\Downarrow$$

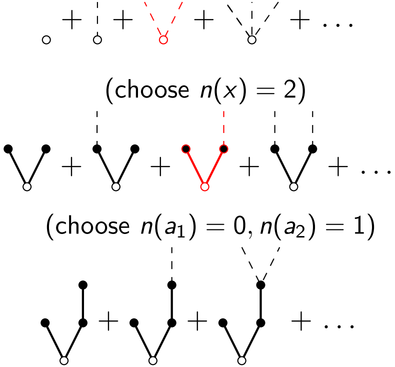
(choose $n(x) = 2$)



$$\Leftrightarrow \quad \frac{1}{z^4} \sum_{a_1 a_2} S_{xa_1} S_{xa_2} \left[ \sum_{n(a_1)} \left( \frac{1}{z^2} \sum_{b} S_{a_1 b}\, u_b \right)^{n(a_1)} \right] \left[ \sum_{n(a_2)} \cdots \right]$$

# Solving the QVE

For  $:= \frac{1}{z^2}\sum_{y=1}^{N} S_{xy}u_y$,  $:= \Big($  $\Big)^2, \ldots,$ we have




$$\Leftrightarrow \quad \sum_{n(x)=0}^{\infty} z^{-2n(x)} \left( \sum_{y} S_{xy} u_y \right)^{n(x)} \overset{\text{(def)}}{=} u_x$$

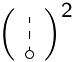
$$\Downarrow$$

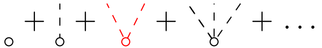
$$\Leftrightarrow \quad \frac{1}{z^4} \sum_{a_1 a_2} S_{xa_1} S_{xa_2} \left[ \sum_{n(a_1)} \left( \frac{1}{z^2} \sum_{b} S_{a_1 b} u_b \right)^{n(a_1)} \right] \left[ \sum_{n(a_2)} \cdots \right]$$

$$\Downarrow$$

$$\Leftrightarrow \quad \frac{1}{z^{2(2+0+1)}} \sum_{a_1 a_2 b} S_{xa_1} S_{xa_2} S_{a_2 b} u_b$$

# Solving the QVE

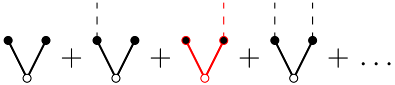
For  $:= \frac{1}{z^2}\sum_{y=1}^{N} S_{xy}u_y$,  $:=$ , ..., we have



$$\Leftrightarrow \quad \sum_{n(x)=0}^{\infty} z^{-2n(x)} \left( \sum_{y} S_{xy}u_y \right)^{n(x)} \overset{\text{(def)}}{=} u_x$$

(choose $n(x) = 2$)

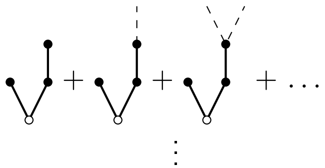
$$\Downarrow$$



$$\Leftrightarrow \quad \frac{1}{z^4}\sum_{a_1 a_2} S_{xa_1}S_{xa_2}\left[\sum_{n(a_1)}\left(\frac{1}{z^2}\sum_{b} S_{a_1 b}u_b\right)^{n(a_1)}\right]\left[\sum_{n(a_2)} \cdots\right]$$

(choose $n(a_1) = 0$, $n(a_2) = 1$)

$$\Downarrow$$

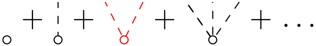


$$\Leftrightarrow \quad \frac{1}{z^6}\sum_{a_1 a_2 b} S_{xa_1}S_{xa_2}S_{a_2 b}u_b$$

$$\vdots$$

# Solving the QVE

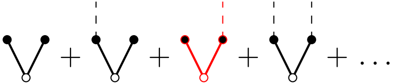
For  $:= \frac{1}{z^2}\sum_{y=1}^{N} S_{xy}u_y$,  $:= \left( \right.$  $\left. \right)^2, \ldots,$ we have



$$\Leftrightarrow \quad \sum_{n(x)=0}^{\infty} z^{-2n(x)} \left( \sum_{y} S_{xy} u_y \right)^{n(x)} \overset{(\text{def})}{=} u_x$$

(choose $n(x) = 2$)

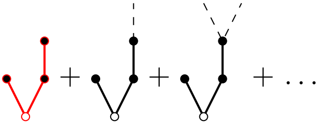
$$\Downarrow$$



$$\Leftrightarrow \quad \frac{1}{z^4} \sum_{a_1 a_2} S_{xa_1} S_{xa_2} \left[ \sum_{n(a_1)} \left( \frac{1}{z^2} \sum_{b} S_{a_1 b} u_b \right)^{n(a_1)} \right] \left[ \sum_{n(a_2)} \cdots \right]$$

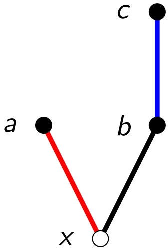
(choose $n(a_1) = 0$, $n(a_2) = 1$)

$$\Downarrow$$

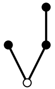


$$\Leftrightarrow \quad \frac{1}{z^6} \sum_{a_1 a_2 b} S_{xa_1} S_{xa_2} S_{a_2 b}$$

# Defining the value of a tree




$\Leftrightarrow$ $$\sum_{a,b,c} S_{xa} S_{xb} S_{bc} = val_x\left( \quad \right)$$



Note:

- sum only over (the labels of) unfilled vertices
- $n$ edges encode factor of $\frac{1}{z^{2n}}$

Formalize this notation by assigning to each rooted, planar tree with $n$ edges $\Gamma \in \mathcal{T}_n$ its value:

$$val_x(\Gamma) := \left( \prod_{v \in V(\Gamma)} \sum_{x_v=1}^{N} \right) \left[ \delta_{x_{\text{root}}=x} \prod_{e \in E(\Gamma)} S_{x_{e_-} x_{e_+}} \right].$$

# Relating trees to DoS

Applying that procedure recursively yields the following **Proposition**:

$$m_x(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\Big[\sum_{\Gamma\in\mathcal{T}_n} val_x(\Gamma)\Big] z^{-2n}. \qquad \square$$

---

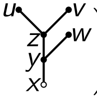
recall: (average of) $m_x$ is the Stieltjes transform of DoS

note: $m_\mu$ Stieltjes transform of $\mu \Rightarrow m_\mu$ analytic in $\mathbb{C} \setminus \operatorname{supp}\mu$

$\Rightarrow$ $\max\operatorname{supp}\rho \leq \limsup_n \left|\sum_{\Gamma\in\mathcal{T}_n} \max_x val_x(\Gamma)\right|^{\frac{1}{2n}}$ by Cauchy-Hadamard

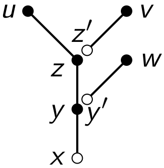
note: cannot bound $val_x(\Gamma)$ unless $\Gamma$ is linear $\Rightarrow val_x(\Gamma) = \sum_i (S^n)_{xi}$

# Bounding $val(\Gamma) := \max_x val(\Gamma)$

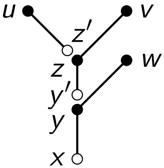
$$val\left( \vphantom{\sum} \right.$$ 


$$\left. \vphantom{\sum} \right) \overset{(\text{def})}{=} \max_x \sum_{yzuvw} S_{xy} S_{yz} S_{zu} S_{zv} S_{yw}$$

$$\leq \max_x \sum_{yzu} S_{xy} S_{yz} S_{zu} \max_{z'} \sum_{v} S_{z'v} \max_{y'} \sum_{w} S_{y'w} \quad \Leftrightarrow$$

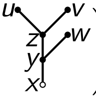



$$\leq \max_x \sum_{yw} S_{xy} S_{yw} \max_{y'} \sum_{zv} S_{y'z} S_{zv} \max_{z'} \sum_{u} S_{z'u} \quad \Leftrightarrow$$

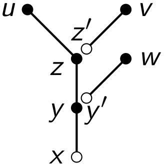



The unfilled vertex indicates we are taking the maximum over this vertex' label instead of summing over it.

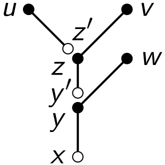
# Bounding $val(\Gamma) := \max_x val(\Gamma)$

$val\Big($  $\Big) \overset{(def)}{=} \max_x \sum_{yzuvw} S_{xy} S_{yz} S_{zu} S_{zv} S_{yw}$



$\leq \|S^3\| \|S\|^2 \quad \Leftrightarrow$  $=:$ "leftmost" splitting

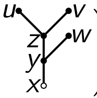


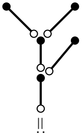
$\leq \|S^2\|^2 \|S\| \quad \Leftrightarrow$  $=:$ "rightmost" splitting



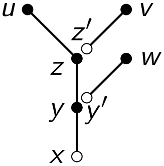
The unfilled vertex indicates we are taking the maximum over this vertex' label instead of summing over it.

# Bounding $val(\Gamma) := \max_x val(\Gamma)$

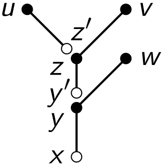
$val\Big($ 

$\Big) \overset{(\text{def})}{=} \max_x \sum_{yzuvw} S_{xy} S_{yz} S_{zu} S_{zv} S_{yw} \leq \|S\|^5 \Leftrightarrow$ 

“complete” splitting

$\leq \|S^3\| \|S\|^2 \quad \Leftrightarrow$ 

$=:$ “leftmost” splitting

$\leq \|S^2\|^2 \|S\| \quad \Leftrightarrow$ 

$=:$ “rightmost” splitting

The unfilled vertex indicates we are taking the maximum over this vertex’ label instead of summing over it.

# Calculating $\sum_{\Gamma \in \mathcal{T}_n} val(\Gamma)$ for fixed $n \gg 1$

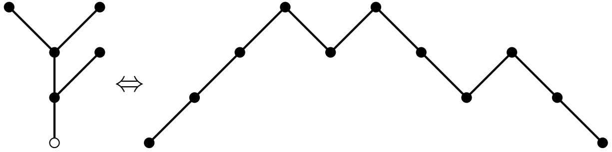
For $\mathbb{E}$ the expectation under the uniform measure on $\mathcal{T}_n$:

$$\sum_{\Gamma \in \mathcal{T}_n} val(\Gamma) = |\mathcal{T}_n| \, \mathbb{E} val(\Gamma)$$

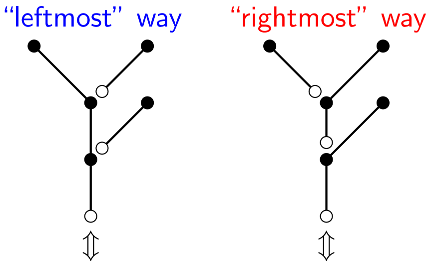
- $\lim_{n \to \infty} |\mathcal{T}_n|^{\frac{1}{2n}} = 2$
- define Dyck paths of length $2n$:

$$D_{2n} := \{\pi : [0, 2n] \cap \mathbb{N} \to \mathbb{N} : \pi(0) = \pi(2n) = 0 \text{ and } |\pi(i) - \pi(i+1)| = 1\}$$

- there is a bijection between Dyck paths $D_{2n}$ and $\mathcal{T}_n$:

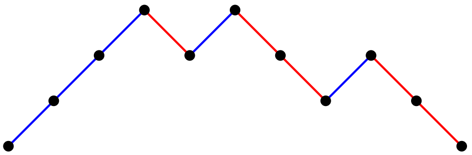


# Comparing bounds obtained by different splittings




$$val(\Gamma) \qquad \leq \|S^3\|\|S\|^2 \qquad \leq \|S^2\|^2\|S\| = \|S\|^5\left(\frac{\|S^2\|}{\|S\|^2}\right)^2$$

Count either up-runs or down-runs. Every $j$-run improves the trivial bound $val(\Gamma) \leq \|S\|^n$ by a factor of $z_j \stackrel{\text{(def)}}{=} \|S^j\|/\|S\|^j \leq 1$.



**Long runs improve our bound** since $\|S^j\| \leq \|S\|^j$ hardly ever saturates and gets better the bigger $j$ by submultiplicativity of $\|\cdot\|$.

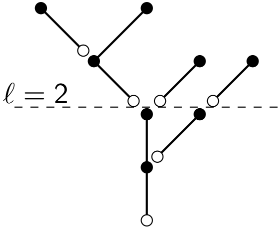
# Bounds obtained by left- and rightmost splittings

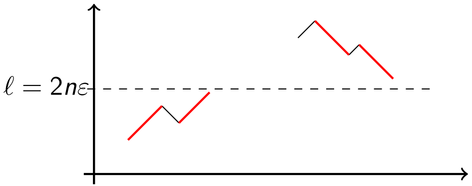
Under this identification of $\mathcal{T}_n \leftrightarrow D_{2n} : \Gamma \mapsto \pi(\Gamma)$, this yields the following bounds:

leftmost: $val(\pi(\Gamma)) \leq \|S\|^n \prod_i z_j^{\#j\text{-up-runs}(\pi(\Gamma))} =: \|S\|^n \, \mathbf{z}^{\mathbf{U}(\pi)}$

rightmost: $val(\pi(\Gamma)) \leq \|S\|^n \prod_i z_j^{\#j\text{-down-runs}(\pi(\Gamma))} =: \|S\|^n \, \mathbf{z}^{\mathbf{D}(\pi)}$

mixture: above height $\ell$ take rightmost splitting, below $\ell$ leftmost:



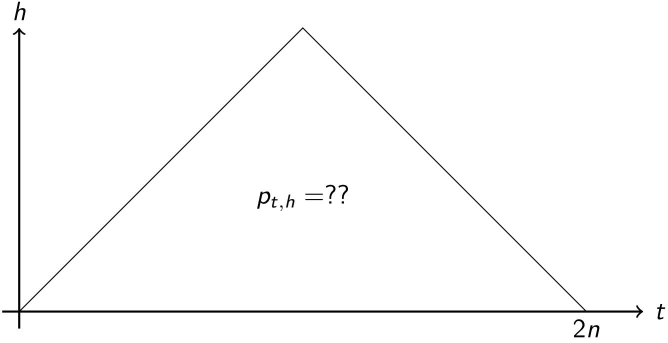




- Write $\|S\|^n \, \mathbf{z}^{\mathbf{T}_\ell(\pi)}$ for the bound obtained by this splitting.
- Have to be careful how to split around height $\ell$, but this can be controlled using the Pigeonhole principle.

# Dyck paths as inhomogeneous Markov chains

Fortunately, [Arnold 1980] showed that the uniform measure over $D_{2n}$ is induced by an inhomogeneous Markov chain with

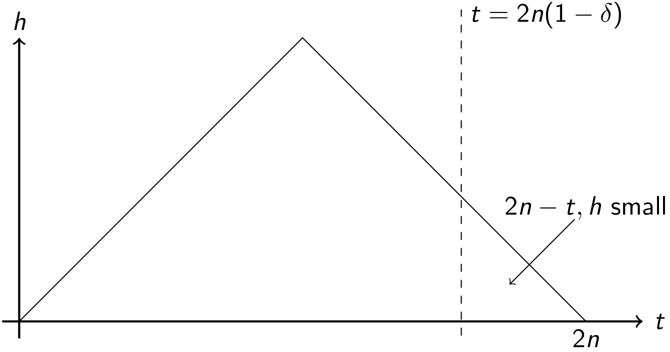
$$p_{t,h} := \mathbb{P}\left(\pi(t+1) = h+1 \,|\, \pi(t) = h\right) = \frac{1}{2}\frac{h+2}{h+1}\frac{2n-t-h}{2n-t}.$$



# Dyck paths as inhomogeneous Markov chains

Fortunately, [Arnold 1980] showed that the uniform measure over $D_{2n}$ is induced by an inhomogeneous Markov chain with

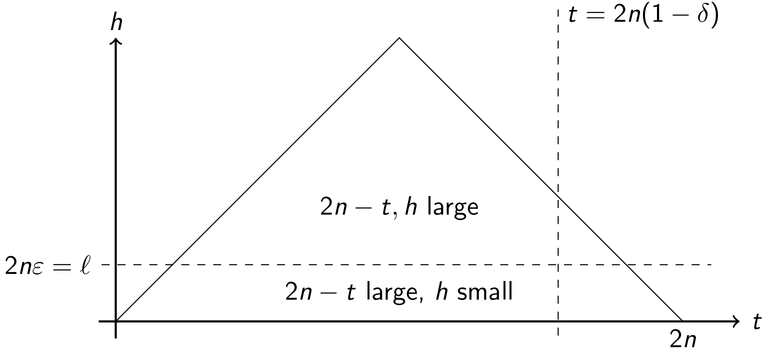
$$p_{t,h} := \mathbb{P}\left(\pi(t+1) = h+1 \,|\, \pi(t) = h\right) = \frac{1}{2}\frac{h+2}{h+1}\frac{2n-t-h}{2n-t}.$$



# Dyck paths as inhomogeneous Markov chains

Fortunately, [Arnold 1980] showed that the uniform measure over $D_{2n}$ is induced by an inhomogeneous Markov chain with

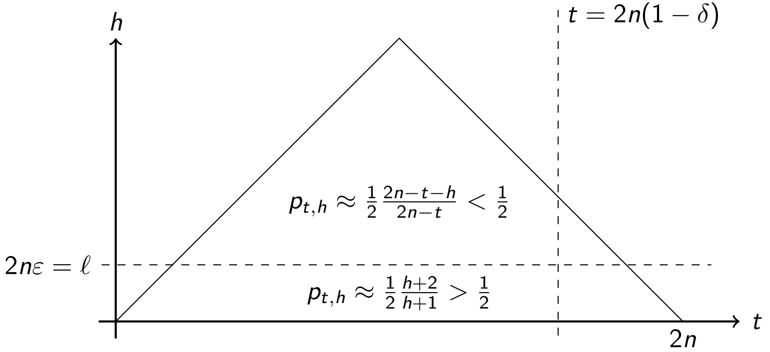
$$p_{t,h} := \mathbb{P}\left(\pi(t+1) = h+1 \,|\, \pi(t) = h\right) = \frac{1}{2}\frac{h+2}{h+1}\frac{2n-t-h}{2n-t}.$$



# Dyck paths as inhomogeneous Markov chains

Fortunately, [Arnold 1980] showed that the uniform measure over $D_{2n}$ is induced by an inhomogeneous Markov chain with

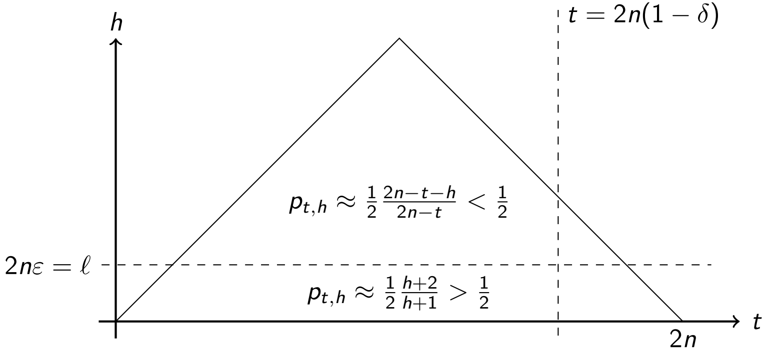
$$p_{t,h} := \mathbb{P}\left(\pi(t+1) = h+1 \,|\, \pi(t) = h\right) = \frac{1}{2}\frac{h+2}{h+1}\frac{2n-t-h}{2n-t}.$$



# Dyck paths as inhomogeneous Markov chains

Fortunately, [Arnold 1980] showed that the uniform measure over $D_{2n}$ is induced by an inhomogeneous Markov chain with

$$p_{t,h} := \mathbb{P}\left(\pi(t+1) = h+1 \mid \pi(t) = h\right) = \frac{1}{2}\frac{h+2}{h+1}\frac{2n-t-h}{2n-t}.$$

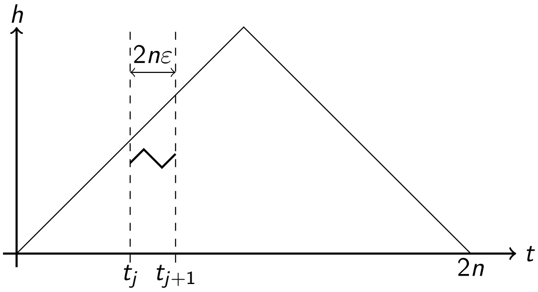


# How to bound $\mathbb{E}val(\pi(\Gamma))$

Under the simple random walk ($\mathbb{E}_{\frac{1}{2}}$) of length $2n \to \infty$ we have the explicit formula (by [Holst 2014] and Cauchy Hadamard):

$$\limsup_{n\to\infty} \mathbb{E}_{\frac{1}{2}} \prod_{i \le J} z_j^{\#j\text{-up-runs}(\pi)} \stackrel{!}{=} w_c(J) \quad \forall J \in \mathbb{N}$$

Markov property allows us to chop up path (in time) in $1/\varepsilon$ "small" segments $[t_j, t_{j+1}] \cap \mathbb{N}$.



# Approximation by (time-)homogeneous Markov chain

- This allows us to approximate $p_{t,h}$ by "nicer" rates $\tilde{p}_{t,h}$ such that
$$\tilde{p}_{t,h} > \frac{1}{2} \text{ if } h \leq \ell \quad \text{and} \quad \tilde{p}_{t,h} < \frac{1}{2} \text{ if } h > \ell.$$
- Recall that $\mathbf{T}_\ell$ counted up-runs if $h \leq \ell$ and down-runs if $h > \ell$.
- Holley's inequality and submultiplicativity of the norm go in the right direction, i.e.

$$\mathbb{E}_p \mathbf{z}^{\mathbf{U}} \leq \mathbb{E}_{\frac{1}{2}} \mathbf{z}^{\mathbf{U}} \quad \text{if } p > \frac{1}{2} \quad \text{and} \quad \mathbb{E}_p \mathbf{z}^{\mathbf{D}} \leq \mathbb{E}_{\frac{1}{2}} \mathbf{z}^{\mathbf{D}} \quad \text{if } p < \frac{1}{2},$$

yielding (after taking $\lim_{\varepsilon,\delta \to 0} \limsup_{n \to \infty}$):

$$_{\text{inverse factor of improvement}} = \mathbb{E}\mathbf{z}^{\mathbf{T}_\ell} \leq \mathbb{E}_{\frac{1}{2}} \mathbf{z}^{\mathbf{T}_\ell} = \mathbb{E}_{\frac{1}{2}} \mathbf{z}^{\mathbf{U}} {}_{= \text{known!}},$$

where $\mathbb{E}$ is expectation under uniform distribution of Dyck paths and $\mathbb{E}_{\frac{1}{2}}$ is expectation under the simple random walk.

# Summary

- We gave an explicit bound on the largest eigenvalue of a general Wigner type matrix in terms of the norm of the matrix of variances $S$ and its powers.
- The solution of the corresponding deteministic equation (QVE) is expressed in terms of the statistics of up-runs and down-runs in the canonical Dyck-path ensemble.
- The inhomogeneous Markov chain behind the Dyck path ensemble can be approximated by a more tractable Markov chain, which, in turn, can be stochastically compared with the simple random walk (Holley inequality).

Thank you for your attention!

# Additional material

# Holley's inequality (generalisation of FKG)

**Setting**

- $\Omega := \{+1, -1\}^E$, with $E$ finite and $\omega \leq \omega'$ if $\omega(i) \leq \omega'(i)\ \forall i$
- $\mathbb{P}_1, \mathbb{P}_2$ probability measures on $\Omega$
- $X$ increasing, i.e. $\omega \leq \omega' \Rightarrow X(\omega) \leq X(\omega')$

**If**

1. $\mathbb{P}_1(\omega^e)\mathbb{P}_2(\omega_e) \leq \mathbb{P}_1(\omega_e)\mathbb{P}_2(\omega^e)$, and
2. for $\mathbb{P}$ being $\mathbb{P}_1$ or $\mathbb{P}_2$ one has $\mathbb{P}(\omega_{ef})\mathbb{P}(\omega^{ef}) \geq \mathbb{P}(\omega_f^e)\mathbb{P}(\omega_e^f)$,

where $\omega^e(t) := 1$, $\omega_e(t) := -1$ if $t = e$ and equal to $\omega$ else.

**Then**

$$\mathbb{E}_1 X \leq \mathbb{E}_2 X$$

**What we have to prove**

- $\mathbf{z}^{\mathbf{D}(\omega_e)} \leq \mathbf{z}^{\mathbf{D}(\omega^e)}$
- $\mathbb{P}_{c<1/2}(\omega^e)\mathbb{P}_{\frac{1}{2}}(\omega_e) \leq \mathbb{P}_{c<1/2}(\omega_e)\mathbb{P}_{\frac{1}{2}}(\omega^e)$

# Holley's inequality (generalisation of FKG)

**Setting**

- $\Omega := \{+1, -1\}^E$, with $E$ finite and $\omega \leq \omega'$ if $\omega(i) \leq \omega'(i)\ \forall i$
- $\mathbb{P}_1, \mathbb{P}_2$ probability measures on $\Omega$
- $X$ increasing, i.e. $\omega \leq \omega' \Rightarrow X(\omega) \leq X(\omega')$

**If**

1. $\mathbb{P}_1(\omega^e)\mathbb{P}_2(\omega_e) \leq \mathbb{P}_1(\omega_e)\mathbb{P}_2(\omega^e)$, and
2. for $\mathbb{P}$ being $\mathbb{P}_1$ or $\mathbb{P}_2$ one has $\mathbb{P}(\omega_{ef})\mathbb{P}(\omega^{ef}) \geq \mathbb{P}(\omega_f^e)\mathbb{P}(\omega_e^f)$,

where $\omega^e(t) := 1$, $\omega_e(t) := -1$ if $t = e$ and equal to $\omega$ else.

**Then**

$$\mathbb{E}_1 X \leq \mathbb{E}_2 X$$

**What we have to prove**

- $\mathbf{z}^{\mathbf{D}(\omega_e)} \leq \mathbf{z}^{\mathbf{D}(\omega^e)}$
- $\mathbb{P}_{c<1/2}(\omega^e) \leq \mathbb{P}_{c<1/2}(\omega_e)$