

# Critical Parameters of Loop and Bernoulli Percolation

Peter Mühlbacher

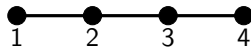
University of Warwick

June 24, 2019

# Setting

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- ▶ Fix  $\beta \in (0, \infty)$ .
- ▶ To each  $e \in E$  assign an independent Poisson point process of unit intensity placing crosses on  $\{e\} \times [0, \beta)$ .

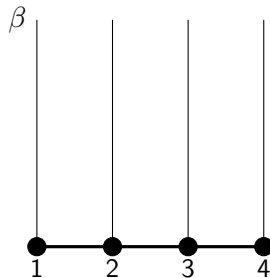
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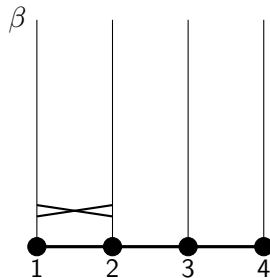
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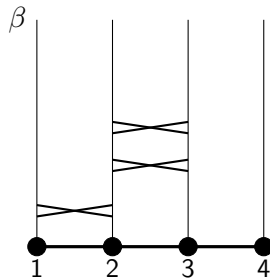
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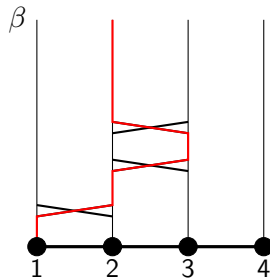
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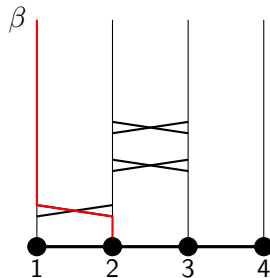
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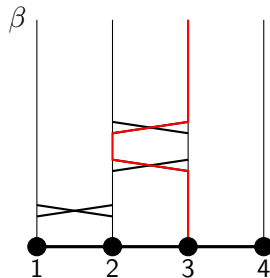
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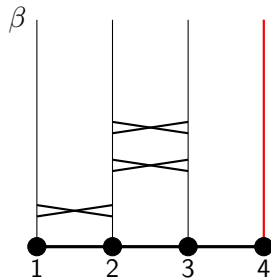
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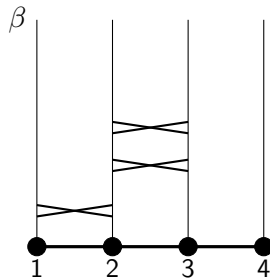
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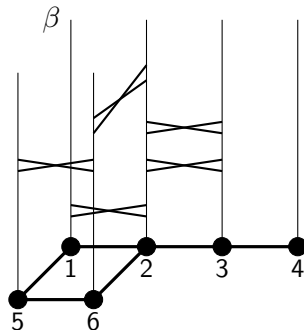


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$$\sigma_\beta = \tau_{12} \circ \tau_{23} \circ \tau_{23} \circ \tau_{56} \circ \tau_{26}$$

# Motivation

- ▶ *Tóth '93*: Lower bound on pressure of the spin- $\frac{1}{2}$  quantum Heisenberg ferromagnet in terms of cycle lengths of random permutations.

*[...] the expected phase transition of the model is closely related to the appearance of an infinite cycle in the random stirring  $\sigma_\beta$  of  $\mathbb{Z}^d$ , for  $\beta$  sufficiently large.*

- ▶ *Aizenman, Nachtergaele '94*: Spin correlations of spin- $\frac{1}{2}$  quantum Heisenberg antiferromagnet in terms of “cycles” of a related model.
- ▶ *Ueltschi '13*: Extension of AN'94 to various other quantum spin models, including quantum XY and quantum ferromagnet.

## Progress so far

### Complete graph:

- ▶ *Schramm '05*: Explicitly calculated joint distribution of normalised cycle lengths. (In particular: Large cycles appear.)
- ▶ *Berestycki '10*: Direct proof for large cycles.
- ▶ *Alon, Kozma '12*: As above, using representation theory.

### $d$ -regular trees:

- ▶ *Angel '03*: Large cycles appear ( $d \geq 5$ )
- ▶ *Hammond '12, '13*: Large cycles appear ( $d \geq 3$ ), more information about when they appear and that they stay.

### Hypercube:

- ▶ *Kotecký, Miłoś, Ueltschi '16*: Large cycles appear.

### Hamming graph:

- ▶ *Miłoś, Şengül '16*: Large cycles appear.
- ▶ *Adamczak, Kotowski, Miłoś '18*: Mean field behaviour for cycle length distribution (and much more).

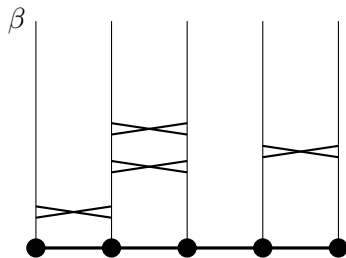
## Coupling with a percolation process

Can there be large cycles on  $G = \mathbb{Z}$ ? I.e. is there a  $\beta$  such that

$$\lim_{L \rightarrow \infty} \mathbb{P}_\beta(\text{size of cycle containing } 0 > L) > 0 ?$$

NO!

Couple the interchange process to a Bernoulli percolation process by throwing away all edges without crosses:



Cycles have to be subsets of percolation clusters!

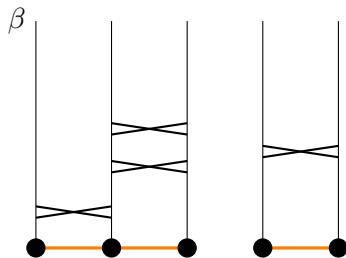
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# Percolation bound

Cycles always have to be subsets of percolation clusters, so we see that for general graphs  $G$

no infinite percolation cluster  $\implies$  no infinite cycles!

Introducing

- ▶  $\beta_c^{\text{cycles}} := \inf\{\beta : \exists \text{ infinite cycle with positive probability}\},$
- ▶  $\beta_c^{\text{perc}} := \inf\{\beta : \exists \text{ infinite percolation cluster with pos. prob.}\},$

one has equivalently:

$$\beta_c^{\text{percolation}} \leq \beta_c^{\text{cycles}}.$$

On sharpness of  $\beta_c^{\text{percolation}} \leq \beta_c^{\text{cycles}}$

$\beta_c^{\text{percolation}} < \beta_c^{\text{cycles}}$  iff there is a  $\beta$  such that

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**YES** on  $d$ -regular trees (by Hammond '12) and in fact on all graphs of bounded vertex degree, e.g.  $\mathbb{Z}^d$ :

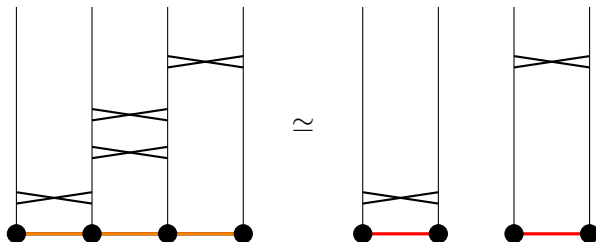
### Theorem (Mühlbacher '19)

*On graphs of uniformly bounded vertex degree one has*

$$\beta_c^{\text{percolation}} < \beta_c^{\text{cycles}}.$$

## Key idea of the proof

Percolation bound is too generous:

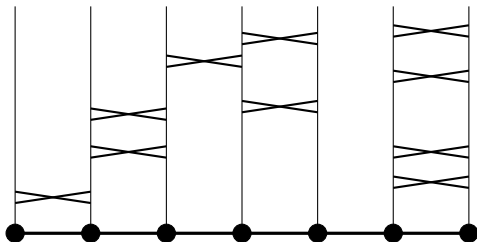


So remove occurrences of such double crosses. If there are “enough”, this will split up infinite clusters into finite ones.

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1. Colour edges with 2 crosses and no crosses in between on adjacent edges in **blue**.
2. Colour remaining edges with  $\geq 1$  crosses **red**.

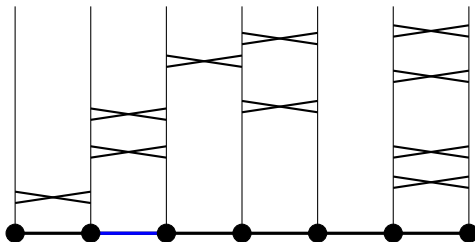
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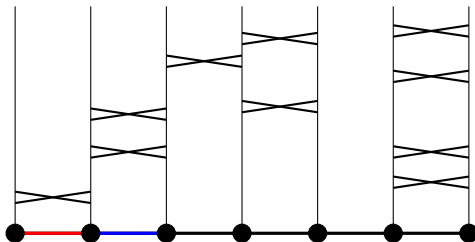
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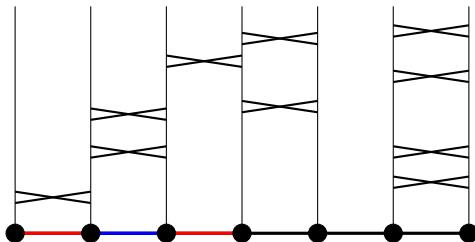
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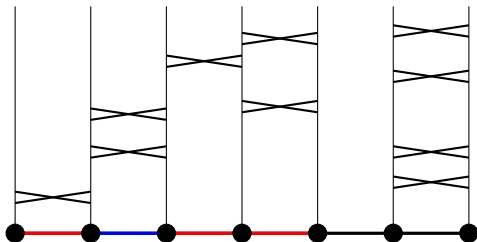
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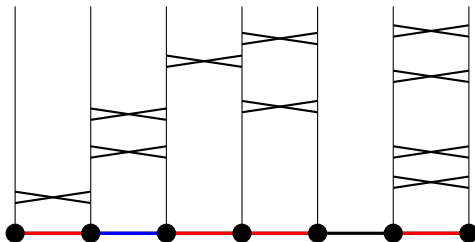
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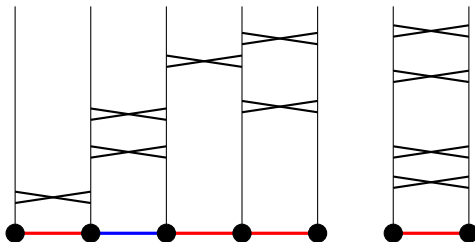
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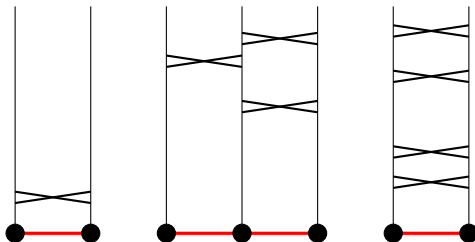
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Naïvely: Take  $\beta = \beta_c^{\text{percolation}} + \varepsilon$  and note that  $\mathbb{P}(e \text{ is blue}) > 0$ .  
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**Solution:** Show that the blue process dominates a product measure.

## A rough outline of the proof

To show that the blue process stochastically dominates a product measure, it suffices<sup>1</sup> to show the finite energy property:

$$\mathbb{P}(e \text{ is blue} | \text{arbitrary colours on edges } \neq e) \geq \textit{const} > 0.$$

To this end we

- ▶ refine the  $\sigma$ -algebra to obtain a spatial Markov property,
- ▶ show that there are only finitely many local configurations that can decrease probability of  $e$  being blue,
- ▶ bound probabilities of these configurations away from 1, so we can “condition them away”.

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<sup>1</sup>Liggett, Schonmann, Stacey '97

# Conclusion

- ▶ Introduced the interchange process.
- ▶ Bounded the critical parameter for existence of infinite cycles in terms of percolation.
- ▶ Original contribution: Showed that, in contrast to graphs of diverging vertex degree, this bound is not sharp on graphs of bounded degree, e.g.  $\mathbb{Z}^d$ .

Thank you for your attention!