

Throughout we differentiate between *loops* (the trajectory following the time-axis and the crosses/bars) and *cycles* (intersection of loops with time = 0). *Giant* means proportional to the volume of the system (i.e. macroscopic and not merely mesoscopic) and is thus only used if we consider an increasing sequence of graphs. *Infinite* is typically used when we consider an infinite graph from the start (or when we show that the probability of return after finite time is not 1, no matter how large we choose the time). *Phase transition* means that some property of a system (e.g. probability of giant cycles, spontaneous magnetisation) undergoes a transition as some parameter (here the inverse temperature β , unless mentioned otherwise) is changed; the smallest β' such that we have some property for all larger β will be referred to as *critical (inverse) temperature*. *Sharpness* of a phase transition is comparable to the cutoff phenomenon. *PD* is a shorthand for Poisson-Dirichlet distribution and unless mentioned otherwise means that the joint distribution of appropriately rescaled cycle lengths converges to PD. Sometimes we weight configurations by $\theta^{\#\text{cycles}}$, sometimes by $\theta^{\#\text{loops}}$.

1. RANDOM INTERCHANGE ($u = \theta = 1$)

Origins: [Harris \[1972\]](#) (foundations of random interchange) and [Tóth \[1993\]](#) (quantum Heisenberg ferromagnet: long range order and spontaneous magnetisation correspond to the appearance of giant cycles in random interchange with $\theta = 2$)

1.1. On complete graph. [Schramm \[2005\]](#) (giant cycles and PD), [Berestycki \[2010\]](#) (easier proof of giant cycles), [Alon and Kozma \[2013\]](#) (states probability of large cycles on general graphs in terms of eigenvalues of Laplacian; uses representation theory), [Berestycki and Kozma \[2012\]](#) (evaluates formula in the former paper on complete graph to show phase transition)

1.2. On trees. [Angel \[2003\]](#) (infinite cycles for $d \geq 5$; asymptotic expansion in d^{-1} for β_c), [Hammond \[2012\]](#) (infinite cycles for $d \geq 3$ and $\beta > \beta_c$ large enough; bound on β_c for $d \geq 55$), [Hammond \[2015\]](#) (phase transition for occurrence of infinite cycles for $d \geq 764$)

1.3. On hypercube. [Kotecký, Miłoś, and Ueltschi \[2016\]](#) (phase transition for giant cycles)

1.4. On Hamming graph. [Miłoś and Şengül \[2016\]](#) (phase transition for giant cycles)

1.5. On $\mathbb{Z}^3$. [Barp, Barp, Briol, and Ueltschi \[2015\]](#) (numerics on critical temperatures for giant cycles and strict inequality for $\mathbb{Z}^3$, as well as PD)

2. RANDOM LOOPS $\theta = 1, u \in [0, 1]$

Origins: [Aizenman and Nachtergaele \[1994\]](#) (quantum Heisenberg antiferromagnet), [Ueltschi \[2013\]](#) (interpolation between ferro- and antiferromagnet; includes XXZ models)

2.1. On complete graph. [Björnberg, Kotowski, Lees, and Miłoś \[2018b\]](#) (convergence of cycle sizes (with bars) to $PD(1/2)$)

2.2. On trees. [Björnberg and Ueltschi \[2018b\]](#) (infinite loops for some critical parameter; for $d \gg 1$, $\beta = O(1/d)$; loop percolation threshold corresponds, to first order in $1/d$, to percolation threshold for occupied edges), [Hammond and Hegde \[2018\]](#) (for $d \geq 56$ infinite cycles for all β sufficiently large; notes that for any connected graph of bounded degree: infinite cycles $\Leftrightarrow$ infinite loops)

3. RANDOM LOOPS WITH $\theta \in \mathbb{R}_+, u \in [0, 1]$

3.1. On complete graph. [Björnberg \[2015\]](#) (for $\theta > 1$ and $\theta^{\#\text{cycles}}$, giant cycles appear), [Björnberg \[2016\]](#) (calculate free energy and critical temperature for $\theta \in \{2, 3, \dots\}$; latter coincides with classical Potts model); [Alon and Kozma \[2018\]](#) (for $\theta^{\#\text{cycles}}$; expression for magnetisation on general graphs; closed-form expression for complete graph implying phase transition), [Björnberg, Fröhlich, and Ueltschi \[2018a\]](#) (for $\theta^{\#\text{loops}}$; phase transitions + spontaneous symmetry breaking; certain characteristic functions coincide with those of PD)

3.2. On trees. [Björnberg and Ueltschi \[2018a\]](#) (for $\theta^{\#\text{loops}}$; asymptotic expansion of critical temperature in d^{-1}), [Betz, Ehlert, and Lees \[2018\]](#) (same as above, but on Galton-Watson trees with sufficiently high offspring expectation; idea: start with $\mathbb{Z}^d$ ($d \gg 1$), delete edges that are unoccupied, end up with random object, assume it's a tree, Galton-Watson distribution for mathematical convenience)

3.3. On Hamming graph. [Adamczak, Kotowski, and Miłoś \[2018\]](#) (for $\theta^{\#\text{cycles}}$, probably also includes $\theta^{\#\text{loops}}$; phase transition for giant cycles and PD)

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