

Contents

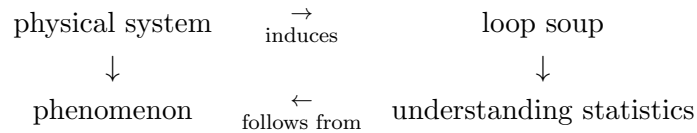
1	Introduction	1
2	Motivation	1
2.1	Feynman '53	1
2.2	Sütő '93 [2]	1
2.3	Tóth '93 [3]	4
2.3.1	Random stirring representation	4
2.3.2	Cycle representation	6
2.4	Symanzik-BFS loop representation of $O(N)$ models [4]	7
2.4.1	Random walk identities	7
2.4.2	Calculating the 2-point function	9

1 Introduction

The material outlined here roughly follows [1], unless mentioned otherwise.

Definition 1.1 (Loop soup). *Probability distribution on multisets of one-dimensional, closed trajectories.*

2 Motivation



At this point it should be noted that this correspondence can also go the other way round; more precisely there are properties of certain loop soup models for which no proof not relying on results from statistical mechanics has been found yet.

2.1 Feynman '53

Lattice permutations. Endow set of permutations on $\Lambda = ([0, L] \cap \mathbb{N})^d$ with probability measure s.t.

$$\mathbb{P}(\sigma) \propto \exp \left\{ -\alpha \sum_{x \in \Lambda} \|x - \sigma(x)\|^2 \right\}.$$

Considering for every permutation σ its set of cycles $C(\sigma)$, this clearly induces a loop soup. It is a simplification of Feynman's representation of the interacting Bose gas.

2.2 Sütő '93 [2]

Occurrence of macroscopic cycles implies Bose-Einstein condensate (in ideal gas) under probability measure on permutations induced by

$$\begin{aligned} & \int_{\Lambda^N} \left(\int_{x_1 \mapsto \sigma(x_1)} \cdots \int_{x_N \mapsto \sigma(x_N)} \left(\exp \left\{ - \sum_{i < j}^N \int_0^\beta U(\omega_i(s) - \omega_j(s)) ds \right\} \right) dW(\omega_1) \dots dW(\omega_N) \right) dx \\ &= \int_{\Lambda^N} \mathbb{E} \left[\exp \left\{ - \sum_{i < j}^N \int_0^\beta U(\omega_i(s) - \omega_j(s)) ds \right\} \mathbf{1}_{\{\omega_i(\beta) = x_{\sigma(i)} \ \forall i\}} \middle| \omega_i(0) = x_i \ \forall i \right] dx. \end{aligned} \tag{2.1} \quad \{\text{eq:feynman}\}$$

The following is not about Sütő's paper, but about motivating the choice of weights for each $\sigma \in S_N$ as implied by (2.1). This is done by deriving the Feynman-Kac formula, which expresses the partition function Z_H corresponding to a very natural Hamiltonian H in terms of a sum (over $\sigma \in S_N$) of evaluations of a non-negative functional mapping $\sigma \in S_N$ to $\mathbb{R}_+$. The evaluations of that functional, up to normalisation, can naturally be interpreted as probabilities of permutations $\sigma \in S_N$.

Now consider the Hamiltonian

$$H_{\Lambda,N} := -\frac{1}{2} \sum_{i=1}^N \Delta_i + \sum_{i < j}^N U(x_i - x_j),$$

where Δ_i is the Laplacian for the i -th boson and U is a multiplication operator; now consider the partition function which is just the trace of $e^{-\beta H_{\Lambda,N}}$ on the symmetric subspace of $L^2(\Lambda^N)$, i.e.

$$\text{Tr}_{L^2(\Lambda^N)} P_{\text{sym}} e^{-\beta H_{\Lambda,N}}.$$

In order to calculate the trace of an integral operator we would like to obtain the integral kernel $A(\cdot, \cdot)$ such that

$$\left[e^{-\beta H_{\Lambda,N}} f \right] (x) = \int_{\Lambda^N} A(x, y) f(y) dy.$$

In the finite dimensional case we can then go on to calculate the trace by evaluating $\sum_x A(x, x)$; in the infinite-dimensional case this is more subtle since the diagonal is a set of measure zero, but it can be done in some interesting cases as shown later.

We obtain this A by applying Trotter's product formula^{*} (and plugging in the definition of $H_{\Lambda,N}$):

$$\left[e^{-\beta H_{\Lambda,N}} f \right] (x) = \lim_{n \rightarrow \infty} \left[\left(e^{\frac{\beta}{n} \sum_i^N \Delta_i} e^{-\frac{\beta}{n} \sum_{i < j}^N U(x_i - x_j)} \right)^n f \right] (x). \quad (2.2) \quad \{\text{eq:trotter}\}$$

Now note that $e^{-\frac{\beta}{n} \sum_{i < j}^N U(x_i - x_j)}$ still acts as a multiplication operator[†], while $e^{\frac{\beta}{n} \sum_i^N \Delta_i}$ acts like the usual heat evolution (independently on every coordinate x_i), i.e.

$$\left[e^{t \sum_i^N \Delta_i} f \right] (x) = \int_{\Lambda^N} g_t(x, y) f(y) dy, \quad g_t(x, y) := \text{const} \cdot \prod_i^N e^{-(x_i - y_i)^2 / 2t}. \quad (2.3) \quad \{\text{eq:heat}\}$$

Repeatedly plugging in (2.3) into (2.2), using that $e^{-\frac{\beta}{n} \sum_{i < j}^N U(x_i - x_j)}$ is a multiplication operator, using dominated convergence, and noting that the Wiener measure is characterized by its finite dimensional marginals, in particular:

$$\int \cdots \int g_{\frac{\beta}{n}} \left(x, y_{\frac{\beta}{n}} \right) \cdots g_{\frac{\beta}{n}} \left(y_{\frac{(n-1)\beta}{n}}, y_{\frac{n\beta}{n}} \right) f \left(y_{\frac{n\beta}{n}} \right) dy_{\frac{\beta}{n}} \cdots dy_{\frac{n\beta}{n}} \rightarrow \int_{\omega: x \mapsto y} f(y) dW(\omega),$$

where dW is the Wiener measure on the space of Brownian bridges between x and y , we arrive at

$$\left[e^{-\beta H_{\Lambda,N}} f \right] (x) = \int_{\Lambda^N} \underbrace{\int_{x_1 \mapsto y_1} \cdots \int_{x_N \mapsto y_N} e^{-\sum_{i < j}^N \int_0^\beta U(\omega_i(s) - \omega_j(s)) ds} dW(\omega_1) \cdots dW(\omega_N)}_{=A(x,y)} f(y) dy. \quad (2.4) \quad \{\text{eq:Axydef}\}$$

Now note that P_{sym} with

$$P_{sym} f(x) = P_{sym} f(x_1, \dots, x_N) := \frac{1}{N!} \sum_{\sigma \in S_N} f(x_{\sigma(1)}, \dots, x_{\sigma(N)}),$$

is the orthogonal projection onto the symmetric subspace of $L^2(\Lambda^N)$. Then if $A(\cdot, \cdot)$ is the integral kernel of an operator $A : L^2(\Lambda^N) \rightarrow L^2(\Lambda^N)$ we have that the integral kernel A_{sym} of $P_{sym} A$ can be written^{*} as $A_{sym}(x, y) = \frac{1}{N!} \sum_{\sigma} A(\sigma(x), y)$.

^{*}Trotter's product formula can be seen as a way to reduce the problem of finding the integral kernel of e^{A+B} to finding the integral kernels of e^A and e^B and convolving them appropriately.

[†]Formally we have: $\left[e^{-\frac{\beta}{n} \sum_{i < j}^N U(\cdot - \cdot)} f \right] (x) = e^{-\frac{\beta}{n} \sum_{i < j}^N U(x_i - x_j)} f(x)$.

^{*}To see this evaluate $[P_{sym} A f](x)$ in two ways: $[P_{sym} A f](x) = [(P_{sym} A) f](x) \stackrel{(\text{def})}{=} \int A_{sym}(x, y) f(y) dy$; now compare with $[P_{sym} [A f](\cdot)](x) = [\frac{1}{N!} \sum_{\sigma} (\int A(\sigma(\cdot), y) f(y) dy)](x) = \int \frac{1}{N!} \sum_{\sigma} A(\sigma(x), y) f(y) dy$. Note that $\sigma(x)$ is a shortcut for $(x_{\sigma(1)}, \dots, x_{\sigma(N)})$.

We extend the formula $Tr K = \sum_x K(x, x)$ (for finite-dimensional K) to positive operators $K \geq 0$ by setting $Tr K = \int \tilde{K}(x, x) dx$, where we defined (pointwise!) $\tilde{K}(x, y) := \int K^{\frac{1}{2}}(x, t) K^{\frac{1}{2}}(t, y) dt$ (this is well-defined as opposed to $\int K(x, x) dx$ which is an integral over the diagonal elements, a measure 0 subset - see [this stack-exchange answer](#)).

Putting the above steps together and assuming $A(x, y)$, as (sloppily) defined in (2.4), satisfies $A(x, y) = \int A^{\frac{1}{2}}(x, t) A^{\frac{1}{2}}(t, y) dt$, we have outlined the proof of

$$Tr_{L^2(\Lambda^N)} P_{sym} e^{-\beta H_{\Lambda, N}} = \frac{1}{N!} \sum_{\sigma \in S_N} \int_{\Lambda^N} A(x, \sigma(x)) dx. \quad (2.5) \quad \{\text{eq:feynman-kac}\}$$

2.3 Tóth '93 [3]

2.3.1 Random stirring representation

Yields an improved lower bound on pressure of 1/2-spin quantum Heisenberg ferromagnet.

Can be derived in the following way for a hard-core, nearest neighbours interaction

$$U(x_i - x_j) = \begin{cases} \infty & \text{if } |x_i - x_j| = 0 \\ -2 & \text{if } |x_i - x_j| = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (2.6) \quad \{\text{eq:hardcoreU}\}$$

By a Feynman-Kac argument (just like the derivation of (2.5), but in a spatially discrete setting), we express* the N -particle partition function $Z = Z_{\beta, N}$ as

$$Z = \frac{1}{N!} \sum_{\sigma \in S_N} \sum_{x_1, \dots, x_N \in \Lambda} \mathbb{E} \left[\exp \left\{ - \int_0^\beta \sum_{i < j} U(\omega_i(s) - \omega_j(s)) ds \right\} \mathbf{1}_{\{\omega_i(\beta) = x_{\sigma(i)}\}} \middle| \omega_i(0) = x_i \forall i \right], \quad (2.7) \quad \{\text{eq:discreteFeynm}\}$$

where $\{\omega_i\}_i$ are N independent, continuous time, simple symmetric random walks with jump rate $2d$; i.e. the duration until the next jump is exponentially distributed with expectation $(2d)^{-1}$. Introducing $\tau := \inf\{s \geq 0 : \exists i \neq j \text{ s.t. } \omega_i(s) = \omega_j(s)\}$,

*This part still holds for arbitrary potentials U .

$\mathcal{B}(A) := |\{(x, y) \in A \times A : |x - y| = 1\}|$ and using (2.6), we can write (2.7) as

$$Z = \sum_{\substack{A \subseteq \Lambda \\ |A|=N}} \mathbb{E} \left[\exp \left\{ \int_0^\beta \mathcal{B}(\{\omega_1(s), \dots, \omega_N(s)\}) ds \right\} \mathbf{1}\{\tau > \beta\} \right. \\ \left. \cdot \mathbf{1}\left\{ \{\omega_1(\beta), \dots, \omega_N(\beta)\} = A \right\} \middle| \left\{ \omega_1(0), \dots, \omega_N(0) \right\} = A \right]. \quad (2.8) \quad \{\text{eq:setRepr}\}$$

Assuming $\{\tau > \beta\}$ (this ensures injectivity and doesn't change anything since if $\tau \leq \beta$ the contribution vanishes anyway), we may, for every fixed s , think of the paths $\{\omega_i(s)\}_i$ as the image of the set $A = \{\omega_i(0)\}_i$ under some permutation σ_s ; so let $\eta_s := \{\sigma_s(x) : x \in A\}$.

What are the dynamics of σ_s as s varies? The exponential clock with rate $2d$ (i.e. expectation $(2d)^{-1}$) can conveniently be interpreted as every adjacent edge having an independent exponential clock with rate (and expectation) 1,^{*} so that we can think of σ_s as a random stirring process (*RSP*), a process starting at $\sigma_0 = id$ and such that a transposition (x, y) is appended to it after exponentially distributed times with rate 1, independently for each pair of nearest neighbours. In order to rewrite (2.8) in terms of the *RSP* (without referring to paths ω_i anymore), we still need to account for the fact that some of the paths will intersect and that we have to disregard these paths (that is what the factor $\mathbf{1}\{\tau > \beta\}$ is there for; note that $\tau = \tau(\{\omega_i\}_i)$). But since the random walks ω_i are isotropic it is easy to see that this corresponds to an additional killing event with rate[†] $\mathcal{B}(\{\omega_1(s), \dots, \omega_N(s)\}) = \mathcal{B}(\eta_s)$. Denote the random event (a stopping time w.r.t. the σ -algebra generated by η_s) with rate $\mathcal{B}(\eta_s)$ at time s by $\tilde{\tau}$. Hence we get that the r.h.s. (2.8) can be rewritten as follows:

^{*}Let X, Y be independent $\text{Exp}(\lambda), \text{Exp}(\mu)$, respectively, then $\mathbb{P}(\min\{X, Y\} \geq t) = \mathbb{P}(X \geq t, Y \geq t) = \mathbb{P}(X \geq t)\mathbb{P}(Y \geq t) = e^{-(\lambda+\mu)t}$, so $\min\{X, Y\}$ is an $\text{Exp}(\lambda + \mu)$ r.v.

[†]The more paths that are close to each other, the more likely it is that two paths intersect.

Expectation in (2.8)

$$\begin{aligned}
&= \mathbb{E} \left[\exp \left\{ \int_0^\beta \mathcal{B}(\eta_s) ds \right\} \mathbf{1}\{\tilde{\tau} > \beta\} \mathbf{1}\{\eta_\beta = A\} \middle| \eta_0 = A \right] \\
&= \mathbb{E} \left[\exp \left\{ \int_0^\beta \mathcal{B}(\eta_s) ds \right\} \mathbb{P}(\tilde{\tau} > \beta | (\eta_s)_{0 \leq s \leq \beta}) \mathbf{1}\{\eta_\beta = A\} \middle| \eta_0 = A \right] \\
&= \mathbb{P}(\eta_\beta = A | \eta_0 = A) = \mathbb{P}(\sigma_\beta(A) = A).
\end{aligned}$$

The first equality follows from the above discussion, the second from the fact that we can pull part of the expectation (by introducing the new stopping time $\tilde{\tau}$ we enlarged the underlying probability space) inside since the other events are independent of it and the third uses the fact that $\mathbb{P}(\tilde{\tau} > \beta | (\eta_s)_{0 \leq s \leq \beta}) = \exp \left\{ - \int_0^\beta \mathcal{B}(\eta_s) ds \right\}$ by definition of $\tilde{\tau}$.[‡] The last equality follows by definition of the random stirring process.

2.3.2 Cycle representation

Consider the *grand canonical partition function* $\Xi_{\beta, \mu} := \sum_{N=0}^{\infty} e^{\beta \mu N} Z_{\beta, N}$. Using the *RSP* representation of $Z_{\beta, N}$ derived above we can write

$$\Xi_{\beta, \mu} = \sum_{A \subseteq \Lambda} e^{\beta \mu |A|} \mathbb{P}(\sigma_\beta(A) = A).$$

Letting $p := \frac{e^{\beta \mu}}{1 + e^{\beta \mu}}$ we get

$$\Xi_{\beta, \mu} = (1 + e^{\beta \mu})^{|\Lambda|} \underbrace{\sum_{A \subseteq \Lambda} p^{|A|} (1 - p)^{|\Lambda \setminus A|} \mathbb{P}(\sigma_\beta(A) = A)}_{(*)}. \quad (2.9) \quad \{\text{eq:newGCPF}\}$$

Now $(*)$ admits the following interpretation: Color every site of Λ independently either red or blue with probability p and $1 - p$, respectively. I.e. extend the probability space so that $p^{|A|} (1 - p)^{|\Lambda \setminus A|} = \mathbb{P}(A \text{ is colored red, } \Lambda \setminus A \text{ is colored blue})$. We

[‡] $\mathbb{P}(\tilde{\tau} > \beta | (\eta_s)_{0 \leq s \leq \beta}) = \prod_t \mathbb{P}(\tilde{\tau} > t + dt | \tilde{\tau} > t, (\eta_s)_{0 \leq s \leq \beta}) = \prod_t e^{-\mathcal{B}(\eta_t) dt}$ in an infinitesimal sense; i.e. the “rate” above is to be understood in the sense of being “locally exponential” with rate $\mathcal{B}(\eta_s)$ at time s .

get

$$\begin{aligned}
(*) &= \sum_{A \subseteq \Lambda} \mathbb{P}(\sigma_\beta(A) = A) \underbrace{\mathbb{P}(A \text{ is colored red, } \Lambda \setminus A \text{ is colored blue})}_{=: C_A} \\
&= \sum_{A \subseteq \Lambda} \mathbb{E}_\sigma \mathbf{1}\{\sigma_\beta(A) = A\} \mathbb{E}_C \mathbf{1}\{C_A\} \\
&= \mathbb{E}_\sigma \mathbb{E}_C \sum_{A \subseteq \Lambda} \mathbf{1}\{\sigma_\beta(A) = A, C_A\} \\
&= \mathbb{E}_\sigma \mathbb{E}_C \sum_{A \subseteq \Lambda} \mathbf{1}\{\text{all cycles of } \sigma_\beta \text{ are unicolor, } C_A\} \\
&= \mathbb{E}_\sigma \mathbb{P}_C(\text{all cycles of } \sigma_\beta \text{ are unicolor}) \\
&= \mathbb{E}_\sigma \prod_{\ell \geq 1} \mathbb{P}(\text{vertices in a cycle of length } \ell \text{ are all colored either red or blue})^{|\{\text{cycles of length } \ell \text{ in } \sigma_\beta\}|} \\
&= \mathbb{E}_\sigma \prod_{\ell \geq 1} \left(p^\ell + (1-p)^\ell \right)^{|\{\text{cycles of length } \ell \text{ in } \sigma_\beta\}|}.
\end{aligned}$$

Plugging this back into (2.9) and using that $\sum_{\ell \geq 1} \ell |\{\text{cycles of length } \ell \text{ in } \sigma_\beta\}| = |\Lambda|$ we see that the factor $(1 + e^{\beta\mu})^{|\Lambda|}$ cancels and we end up with

$$\Xi_{\beta,\mu} = \mathbb{E}_\sigma \prod_{\ell \geq 1} \left(1 + e^{\beta\mu\ell} \right)^{|\{\text{cycles of length } \ell \text{ in } \sigma_\beta\}|}.$$

2.4 Symanzik-BFS loop representation of $O(N)$ models [4]

Assume we have some symmetric pair interaction J (i.e. $J_{xy} = J_{yx}$), symmetric potentials U_x bounded from below (i.e. $U_x(s) = U_x(-s) \geq 0 \forall s, x$) and the probability of a state $\phi = (\phi_x)_{x \in \Lambda}$ is given by

$$\begin{aligned}
d\mathbb{P}(\phi) &\propto e^{\frac{1}{2} \sum_{xy} \phi_x \phi_y J_{xy} - \sum_x U_x(\phi_x)} d\phi \\
&\propto e^{\frac{1}{2} \langle \phi, J \phi \rangle} \prod_x g_x(\phi_x^2) d\phi.
\end{aligned}$$

The existence of such g_x is guaranteed by our assumptions on U_x . As usual we define the normalisation constant/partition function Z and the 2-point function $\langle \phi_x \phi_y \rangle$:

$$Z := \int e^{\frac{1}{2} \langle \phi, J \phi \rangle} \prod_x g_x(\phi_x^2) d\phi \quad \text{and} \quad \langle \phi_x \phi_y \rangle := Z^{-1} \int \phi_x \phi_y e^{\frac{1}{2} \langle \phi, J \phi \rangle} \prod_x g_x(\phi_x^2) d\phi.$$

Note that, apart from $\prod_x g_x(\phi_x^2)$, Z looks like a Gaussian integral and similarly $\langle \phi_x \phi_y \rangle$ looks like the covariance $\mathbb{E} v_x v_y$ of some centered Gaussian v with covariance J^{-1} .

Now recall, e.g. from Wick's theorem^{*} or by direct computation, that for some positive definite matrix $A > 0$ we have

$$\tilde{Z} := \int e^{-\frac{1}{2}\langle \phi, A \phi \rangle} d\phi = \sqrt{\frac{(2\pi)^n}{\det A}} \quad \text{and} \quad \frac{1}{\tilde{Z}} \int \phi_x \phi_y e^{-\frac{1}{2}\langle \phi, A \phi \rangle} d\phi = (A^{-1})_{xy}.$$

Hence we may be tempted to find a random walk interpretation of $\det A^{-1}$ and $(A^{-1})_{xy}$.

2.4.1 Random walk identities

In this section A will denote some strictly diagonally dominated matrix.

{lem:RWInv}

Lemma 2.1 (Random walk interpretation of $(A^{-1})_{xy}$). *Let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ be a diagonal matrix with $\lambda_i \neq 0 \forall i$ and J be a matrix such that $\Lambda - J$ is strictly diagonally dominant[†] and*

$$\begin{aligned} J_{xy} = J_{yx} &\geq 0 \text{ if } x \neq y \\ &= 0 \text{ if } x = y. \end{aligned}$$

If the r.h.s. converges absolutely, the matrix inverse $(\Lambda - J)^{-1}$ exists and is given by

$$((\Lambda - J)^{-1})_{xy} = \sum_{\omega: x \rightarrow y} \left(\prod_{(i,j) \in \omega} J_{ij} \right) \prod_{a=1}^n \lambda_a^{-n(a,\omega)},$$

where the sum goes over all paths^{*} from x to y and $n(a,\omega)$ denotes the number of times ω visited a .

Proof. Since $\Lambda - J$ is strictly diagonally dominant we can expand the inverse in a Neumann series, i.e.

$$(\Lambda - J)^{-1} = \Lambda^{-1} + \Lambda^{-1} J \Lambda^{-1} + \Lambda^{-1} J \Lambda^{-1} J \Lambda^{-1} + \dots$$

^{*}For an accessible proof, see e.g. <https://math.stackexchange.com/a/126767>.

[†]One may also introduce “cemetaries/killing rates” to turn a non diagonally dominant matrix into a diagonally dominant one, see e.g. [5] for this and many more functional integral representations of random walks, but we will not need this here.

^{*}A path ω of length $|\omega| = m$ from x to y is a sequence $((x, i_1), (i_1, i_2), \dots, (i_{m-1}, y))$ with $i_j \in \{1, \dots, n\}$ for all j .

Now consider for example the summand $\Lambda^{-1}J\Lambda^{-1}J\Lambda^{-1}$:

$$\begin{aligned} (\Lambda^{-1}J\Lambda^{-1}J\Lambda^{-1})_{xy} &= \sum_{\substack{a,b,c=1 \\ a=x, c=y}}^n \lambda_a^{-1} J_{ab} \lambda_b^{-1} J_{bc} \lambda_c^{-1} \\ &= \sum_{\substack{\omega: x \rightarrow y \\ |\omega|=2}} \left(\prod_{(i,j) \in \omega} J_{ij} \right) \prod_{a=1}^n \lambda_a^{-n(a,\omega)} \end{aligned}$$

The claim easily follows. $\square$

The random walk picture makes sense in case J_{ij}/λ_i is real; then the strict diagonal dominance ensures that it can be interpreted as a random walk on $\{1, \dots, n\}$ with transition probabilities from i to j given by J_{ij}/λ_i .

$\{\text{lem:RWdet}\}$

Lemma 2.2 (Random walk interpretation of $\det A^{-1}$). *Let Λ, J be as in Lemma 2.1. Then we have*

$$\det(\Lambda - J)^{-1} = \det \Lambda^{-1} \exp \left\{ - \sum_{\tilde{\omega}} \left(\prod_{(i,j) \in \tilde{\omega}} J_{ij} \right) \prod_{a=1}^n \lambda_a^{-n(a,\tilde{\omega})} \right\},$$

where the sum is over all unrooted loops $\tilde{\omega}$, i.e. we only count each equivalence class of loops exactly once. Two loops $\tilde{\omega}$ and $\tilde{\omega}'$ of the same length m are equivalent if there is some t such that $\tilde{\omega}_{i+t} = \tilde{\omega}'_i$ for all i , where $i+t$ is modulo the length m .

Proof.

$$\begin{aligned} \det \Lambda \det(\Lambda - J)^{-1} &= \det(1 - \Lambda^{-1}J)^{-1} \\ &= \exp \left\{ -\text{tr} \ln(1 - \Lambda^{-1}J) \right\} \\ &= \exp \left\{ - \sum_{k=1}^{\infty} \frac{1}{k} \text{tr} (\Lambda^{-1}J)^k \right\} \\ &= \exp \left\{ - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{x=1}^n \sum_{\substack{\omega: x \rightarrow x \\ |\omega|=k}} \left(\prod_{(i,j) \in \omega} J_{ij} \right) \prod_{a=1}^n \lambda_a^{-n(a,\omega)} \right\} \end{aligned}$$

In the last step we used that $\text{tr} A^k = \sum_{i_1, \dots, i_k} A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_k i_1}$ and the same idea from the last proof. Note that every loop of length k is taken into account k times (once for each vertex) and every time its contribution is the same since the edges visited are the same and order does not matter; but because of the $\frac{1}{k}$ from the Taylor

expansion of $\ln(1 - \cdot)$ we can simply interpret this sum as sum over unrooted loops $\tilde{\omega}$ so that we continue with

$$= \exp \left\{ - \sum_{\tilde{\omega}} \left(\prod_{(i,j) \in \tilde{\omega}} J_{ij} \right) \prod_{a=1}^n \lambda_a^{-n(a, \tilde{\omega})} \right\}.$$

□

2.4.2 Calculating the 2-point function

We want to write the 2-point function as a Gaussian integral, so we need to absorb the ϕ dependence of $\prod_x g_x(\phi_x^2)$ into $e^{\frac{1}{2}\langle \phi, J \phi \rangle}$. To this end we plug in its Fourier transform as follows:

$$\begin{aligned} \langle \phi_x \phi_y \rangle &= Z^{-1} \int \phi_x \phi_y e^{\frac{1}{2}\langle \phi, J \phi \rangle} \prod_x \int e^{-ik_x \phi_x^2} \hat{g}_x(k_x) dk_x d\phi \\ &= Z^{-1} \int \phi_x \phi_y e^{\frac{1}{2}\langle \phi, J \phi \rangle} \int \underbrace{e^{-i \sum_x k_x \phi_x^2}}_{e^{-i\langle \phi, \text{diag}(k) \phi \rangle}} \prod_x \hat{g}_x(k_x) dk_x d\phi \\ &= Z^{-1} \int \int \phi_x \phi_y e^{-\frac{1}{2}\langle \phi, (2ik - J) \phi \rangle} \prod_x \hat{g}_x(k_x) dk_x d\phi \\ &= Z^{-1} \int \int \phi_x \phi_y e^{-\frac{1}{2}\langle \phi, (2ik - J) \phi \rangle} d\phi \prod_x \hat{g}_x(k_x) dk_x \end{aligned}$$

Note that in the last step we interchanged order of integration! This is justified by the following observations:

1. We can move the contours (of dk_x) along which we integrate from $\int_{-\infty}^{\infty}$ to $\int_{-\infty - iv}^{\infty - iv}$ for any $v \in \mathbb{R}$; this still gives the same result by Cauchy's integral theorem^{*} as

- our integrand is analytic (so that there are no residues to consider) because we assume that all g_x are smooth and, along with all their derivatives, vanish faster than exponentially (i.e. $g^{(n)}(\cdot)e^{a|\cdot|} \in L^1$ for all $n \in \mathbb{N}$ and $a > 0$) - hence[†] all $\hat{g}_x$ vanish faster than any polynomial $\rightarrow$ the integrals $\int_R^{R-iv} e^{-ik_x \phi_x^2} \hat{g}(k_x) dk_x$ (and similarly $\int_{-R-iv}^{-R}$) go to zero as $R \rightarrow \infty$

^{*}See e.g. [this Wikipedia example on contour integration](#) or [that stackexchange answer](#).

[†]<https://math.stackexchange.com/questions/206362/smoothness-and-decay-property-of-fourier-transformati>

- and we take the rectangular contour from $-R$ to R to $R - iv$ to $-R - iv$ to $-R$.
2. Set $A := 2ik - J$; now for v sufficiently large the real part $\operatorname{Re}(A) = \operatorname{Re}(2i(u_x - iv)_x) - J = v - J$ (every $k_x = u_x - iv$ goes from $-\infty - iv$ to $\infty - iv$) will be positive definite since J is some bounded operator. Writing $A = \operatorname{Re}(A) + 2iu$, we see that $2iu$ only gives some negligible complex phase

$$|e^{-\langle \phi, A \phi \rangle}| = \exp(\underbrace{-\langle \phi, (v - J) \phi \rangle}_{> \operatorname{const}(v) > 0}) \underbrace{|e^{2i \sum_x \phi_x^2 u_x}|}_{=1}.$$

Hence the Gaussian integral converges despite A not being positive definite!

3. Thus the inner integral (and because of the weights $\hat{g}$ also the outer integral) in $\int \int \phi_x \phi_y e^{-\frac{1}{2} \langle \phi, (2ik - J) \phi \rangle} d\phi \prod_x \hat{g}_x(k_x) dk_x$ converges absolutely (note that the result of the inner integral can be bounded uniformly in k_x), so that we can apply [Fubini-Tonelli](#) to justify the interchange of order of integration.

Now we can go on to evaluate the inner ϕ integral $\int \phi_x \phi_y e^{-\frac{1}{2} \langle \phi, (2ik - J) \phi \rangle} d\phi$ by virtue of exercise 4(b) in [6]. See [here](#) for a proof via the [identity theorem](#). We finally arrive at (leaving out the normalisation factor Z^{-1})

$$\begin{aligned} \int \int \phi_x \phi_y e^{-\frac{1}{2} \langle \phi, (2ik - J) \phi \rangle} d\phi \prod_x \hat{g}_x(k_x) dk_x &= \int \underbrace{\sqrt{\frac{(2\pi)^n}{\det A}} A_{xy}^{-1}}_{= \int e^{-\frac{1}{2} \langle \phi, A \phi \rangle} d\phi (A^{-1})_{xy}} \prod_x \hat{g}_x(k_x) dk_x \\ &= \int \int ((2ik - J)^{-1})_{xy} e^{-\frac{1}{2} \langle \phi, (2ik - J) \phi \rangle} d\phi \prod_x \hat{g}_x(k_x) dk_x \\ &= \int \int ((2ik - J)^{-1})_{xy} e^{\frac{1}{2} \langle \phi, J \phi \rangle} d\phi \prod_x e^{-2ik_x} \hat{g}_x(k_x) dk_x, \end{aligned}$$

to which we can apply Lemma 2.1 and note that this gives an additional phase so that we end up with basically the same expression as before. Applying Lemma 2.2 to rewrite the determinant in terms of loops and Taylor expanding the exponentials we arrive at Symanzik's polymer representation. It takes quite a bit of new notation, but mathematically nothing interesting happens anymore - so we simply refer to Theorem 2.1 of [4].

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