

Prerequisites: We will keep referring to (without introducing it properly) the CTRW. See [2] (or [11]) for an introduction and a rigorous definition. This text essentially tries to generalise methods from electrical networks (typically applied in the SRW setting) to the more general setting of CTRWs. References for the electrical network results will be provided in the text – most importantly [6], but [12] offers a good, concise introduction. Only few positive results and several counterexamples why a naive generalisation must fail are found at this point.

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1 Introduction

1.1 Motivation

Consider two types of random walks on graphs $G = (V, E)$: The *simple random walk* SRW and the *cyclic time random walk* CTRW $_{\beta}$ with parameter $\beta \in (0, \infty)$. Note that CTRW $_{\infty}$ can be identified with the SRW. We are interested in whether our random walks are transient, depending on the geometry of the graph G . One popular method to show transience of the SRW on G is to throw away edges of G (in a careful manner) to obtain a “nice” subtree G' of G . By Rayleigh’s monotonicity principle this can only decrease the probability of escaping to the boundary (in case G is finite) or to infinity (in case G is infinite).^{*} For trees it is relatively easy to determine whether the SRW is transient since we can calculate the effective resistance and if it is, this implies (by Rayleigh’s monotonicity principle) that the SRW on G is transient.

It can be seen, for example, that $\mathbb{Z}^3$ admits a subtree on which the SRW is transient, implying by monotonicity that $\mathbb{Z}^3$ itself is transient. This can be extended to higher dimensions by writing $\mathbb{Z}^d$ with $d > 3$ as $\mathbb{Z}^3 \times \mathbb{Z}^{d-3}$ (with the appropriate graph product) and noting that $\mathbb{Z}^3$ can be obtained by removing edges, hence establishing transience for $\mathbb{Z}^d, d \geq 3$. A carefully chosen subtree of the random infinite percolation cluster on $\mathbb{Z}^3$ can be used to show [7] that a SRW on the infinite percolation cluster on $\mathbb{Z}^3$ is transient whenever there is an infinite percolation cluster almost surely.

The CTRW is relatively well understood on trees — in particular sufficient conditions on the average vertex offspring so that it becomes transient are known [2, 8] — so it is natural to ask whether Rayleigh’s monotonicity principle remains valid for CTRWs, i.e. if $\beta < \infty$. Here we want to show that this is not the case in general by providing an explicit counterexample, derive from this example sufficient conditions for cases where it has to fail and conjecture that a weak version of Rayleigh’s monotonicity property still holds under appropriate conditions. We then argue that (unfortunately) this will still not be enough to naively mimic the proof[†] of the SRW on $\mathbb{Z}^3$ being transient to show that CTRW $_{\beta}$ is transient. In fact we argue that one has to work harder for any amenable graph and more generally graphs with Cheeger constant less than 1.[‡]

^{*}Clearly obtaining uniform bounds on the finite case implies the infinite case; generally speaking we can go from one setting to the other depending on what is more convenient without having to worry too much.

[†]There are many proofs for this fact. Here we refer to the one where one chooses a subtree of $\mathbb{Z}^3$ and shows that this subtree is transient.

[‡]The idea is that we would like to embed a 3-regular tree, i.e. one where each vertex has at least 2 offspring since in that case [8] showed that there are infinite loops; now note that a d -regular tree has Cheeger constant $d - 2$. In particular the 2-regular tree, i.e. $\mathbb{Z}$, has Cheeger constant 0 and is hence amenable, just like any $\mathbb{Z}^{d'}$, $d' \in \mathbb{N}$. Note also [4], which proves that every graph G with Cheeger constant $\geq k$ has a spanning forest where each component is isomorphic to some tree with Cheeger constant $= k$. More on that can be found in Subsubsection 3.3.2.

1.2 Notation

Motivated by the identification of CTRW_∞ as SRW, we provide a basic dictionary between the language of electric networks as typically used for proving these results for the SRW and that of CTRWs. We say the function $C : V \times V \rightarrow [0, \infty]$ is a *conductance* if it is symmetric, i.e. $C_{xy} = C_{yx}$, and satisfies $C_{xx} = 0$ for all $x, y \in V$.

Fix some countable set of points V with conductances C and introduce the set of unoriented edges $E := \{\{x, y\} : x \neq y \in V, C_{xy} > 0\}$. Call this *the graph (V, E) induced by the conductances C* . Throughout the text we will assume the following:

Assumption 1.1. *Assume that all graphs (also those induced by conductances) are connected and of bounded degree.*

Choose furthermore some distinguished vertex $a \in V$, henceforth referred to as *origin* or *source*.

Instead of considering the SRW directly, it is instructive to construct a natural coupling between the SRW and the CTRW_β , both with with conductances C . Recall that the transition probabilities P_{xy} of this SRW are given by $P_{xy} = C_{xy}/C_x$, where $C_x := \sum_y C_{xy}$.

Fix some $\beta > 0$. Consider now the following random process: To each edge $e \in E$ attach an independent Poisson point process on $[0, \beta)$ of intensity C_e .^{*} We refer to these points (which are subsets of $E \times [0, \beta)$) as *crosses*. We construct $\text{CTRW}_\beta(C)$, the *cyclic time random walk with conductances C* , in the usual way, see e.g. [11, p.8ff].

The usual CTRW_β , e.g. as introduced in [2], on the graph $\tilde{G} = (V, \tilde{E})$ is recovered by setting $C_e = \mathbf{1}\{e \in \tilde{E}\}$ for all $e \in E$. The SRW on V with conductances C is obtained by projecting $\text{CTRW}_\infty(C)$ on the space-coordinate.

There are multiple equivalent definitions of what it means for a random walk to be transient. Here we pick one that is natural to extend to processes that fail to be Markovian[†]: Informally speaking, we call a random walk transient if there is a positive probability of never returning to its origin. In particular this definition is independent of the origin if the random walk is Markovian and, once started at any one vertex, has a positive probability to go to any other vertex. For non-Markovian processes independence of the transience property of the origin is not as trivial and in fact not true in general.

For $X_t \in V \times [0, \beta)$ an instance of $\text{CTRW}_\beta(C)$ starting at the origin a and any $b \in V \setminus \{a\}$ we introduce the hitting times

$$\tau_b := \inf\{t \geq 0 : X_t = (b, s) \text{ for some } s \in [0, \beta)\}, \quad (1.1) \quad \{\text{eq:default}\}$$

and

$$\tau_a^+ := \inf\{t \geq \min_{x \neq a} \tau_x : X_t = (a, s) \text{ for some } s \in [0, \beta)\}. \quad (1.2) \quad \{\text{eq:default}\}$$

^{*}By symmetry $C_e := C_{xy}$ with $e = \{x, y\}$ is well defined.

[†]“Unfortunately” this is the case here. Note that $\text{CTRW}_\beta(C)$ is Markov up until the first time it hits $V \times \{\beta\}$ – afterwards it starts at $V \times \{0\}$ again, but this time some crosses are already explored and are “not random anymore”. Clearly this never happens for $\text{CTRW}_\infty(C)$ (and it shouldn’t since the usual random walk is a reversible Markov process), but it does happen after finite time for $\text{CTRW}_\beta(C)$ with $\beta < \infty$. Thus in this case the transition probabilities P_{xy} are not well defined as they depend on how we got to x in the first place.

Definition 1.2 (Transience). *The $CTRW_\beta(C)$ is called transient if $\mathbb{P}(\tau_a^+ < \infty) < 1$, i.e. there is a positive chance of never returning to the (space-)origin.*

The definition is chosen in such a way that it coincides with the one for the usual random walk (i.e. $\beta = \infty$). Well-known properties like the following also carry over to our setting:

Lemma 1.3. *If $CTRW_\beta(C)$ on V is transient when started on one vertex $a \in V$ and Assumption 1.1 holds, then it is transient when started on any other vertex $a' \in V$ as well.*

Note that in terms of loops it might have been more natural to define the return time as $\tau_{a,0}^+ := \inf\{t > 0 : X_t = (a, 0)\}$, the first time $(a, 0)$ (instead of any (a, s) for any s) is visited since this closes the loop and the connection between transience and infinite loops becomes obvious.

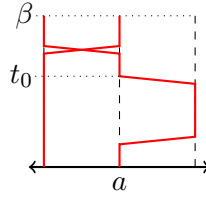


Figure 1: An example to illustrate that a.s. $\tau_a^+ < \tau_{a,0}^+$. Here $\tau_a^+ = t_0 < \beta$, while $\tau_{a,0}^+ = 2\beta$.

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More precisely, following [2], we observe that the following statements are equivalent for finite β :

1. For $CTRW_\beta(C)$ we have $\mathbb{P}(\tau_{a,0}^+ < \infty) = 1$.*
2. The $CTRW_\beta(C)$ a.s. returns infinitely often to the origin.
3. The $CTRW_\beta(C)$ a.s. visits only finitely many vertices.
4. The $CTRW_\beta(C)$ path a.s. has period $k\beta$ for some $k \in \mathbb{N}$.
5. The permutation $\sigma_\beta(C)$ obtained considering the naturally coupled random interchange process a.s. has only finite (permutation-)cycles.

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Note, however, that $\tau_a^+ < \tau_{a,0}^+$ a.s. (see Figure 1) and thus $\mathbb{P}(\tau_a^+ < \infty) < 1 \implies \mathbb{P}(\tau_{a,0}^+ < \infty) < 1$ so our notion of transience is stronger than (i.e. implies) the one introduced in [2] (i.e. with τ_a^+ replaced by $\tau_{a,0}^+$) and in particular we still have that our notion of transience implies existence of infinite loops (the negation of point (3) above). We conjecture that the converse holds as well:

Conjecture 1.4. *Assume Assumption 1.1 holds and that $\beta, C_a < \infty$. If $CTRW_\beta(C)$ on V visits infinitely many vertices before $\tau_{a,0}^+$ with positive probability, then it visits infinitely many vertices before τ_a^+ with positive probability.*

In particular the two notions of transience, i.e. Angel's ($\mathbb{P}(\tau_{a,0}^+ < \infty) < 1$) and ours ($\mathbb{P}(\tau_a^+ < \infty) < 1$), are equivalent.

*This is called recurrent in [2]; transience is then defined as not being recurrent.

The idea is that a positive fraction of paths with $\tau_{a,0}^+ = \infty$ do not return to a at all. One could condition on configurations in some box surrounding the origin a such that there is a positive chance of these paths to never return. In fact this has been done (cf. [2]) on trees for specific values of β .

2 Electric network analogues

2.1 Escape probabilities and Rayleigh's monotonicity principle (RMP)

Methods from electric network theory are particularly well-suited to analyse quantities like the *escape probability* $p_{\text{esc}} = p_{\text{esc}}(a, b)$, which, for an arbitrary random walk on V with conductances C started at a , is defined as the probability of reaching b before returning to a . Here we set

$$p_{\text{esc}} = p_{\text{esc}}(a, b; V, C) := \mathbb{P}(\tau_b < \tau_a^+). \quad (2.1)$$

We can now state what we call Rayleigh's monotonicity principle (RMP):*

Proposition 2.1 (Rayleigh's monotonicity principle). *Fix any V and $a \neq b \in V$. If $C(e) \leq C'(e)$ for all $e = \{x, y\}$ with $x, y \in V$ and equality for e of the form $\{a, x\}$ with $x \in V$, then*

$$p_{\text{esc}}(a, b; V, C) \leq p_{\text{esc}}(a, b; V, C'),$$

where the underlying random walk is $CTRW_\infty(C)$, i.e. the usual random walk with conductances C .

Usually this is proved with Thompson's principle for flows, but a direct and purely probabilistic proof (that seems more likely to be generalisable to our non-Markovian setting) can be found in Appendix A. In particular we have that adding edges (that are not adjacent to the source) cannot decrease the escape probabilities. The assumption that added edges must not be adjacent to the source is not as technical as it may seem at first: Consider $\overset{a}{\circ} \text{---} \overset{b}{\circ} \Rightarrow (-) \overset{c}{\circ} \text{---} \overset{a}{\circ} \text{---} \overset{b}{\circ}$ or more formally: $V = \{a, b, c\}$ with $C_{ab} = 1, C_{ac} = 0, C'_{ab} = 1 = C'_{ac}$ and $C_{bc} = C'_{bc} = 0$. Then the escape probability (w.r.t. C and C' respectively) goes from 1 to 1/2 even though $C \leq C'$.

Note that we did not assume Assumption 1.1 on purpose; in fact this allows the following equivalent interpretation: Starting with a graph and removing edges one by one in an arbitrary order cannot increase the escape probability at any step and thus not as a whole.

2.2 Transience

To relate Rayleigh's monotonicity principle to the transience property we would like to pick " $b = \infty$ "; this is usually (see e.g. [6, p.70ff] or [12]) done as follows: For every $n \in \mathbb{N}$ we consider the graph exhaustion $V^{(n)} := \{x \in V : d(a, x) \leq n\} \cup \{z_n\}$, where d is the graph distance of the graph induced by the conductances. Its edges are induced by the conductances $C^{(n)}$, where we set $C_{xy}^{(n)} = C_{xy}$ whenever both x, y are in $V^{(n)}$ and $C_{x,z_n}^{(n)} = \sum_{z \in V \setminus V^{(n)}} C(x, z)$. Informally

*For a proof see [6, p.54] or combine [12, Proposition 2.5 and Theorem 2.11].

speaking we “glue together” all vertices outside the n -ball around the origin a , i.e. $V^{(n)}$ can also be seen as V with infinite conductance between all $x, y \in V \setminus V^{(n)}$. Then it makes sense to define

$$p_{\text{esc}}(a, \infty; V, C) := \lim_{n \rightarrow \infty} p_{\text{esc}}(a, z_n; V^{(n)}, C^{(n)}). \quad (2.2) \quad \{\text{eq:PescT}$$

For random walks that are reversible and Markovian (e.g. $\text{CTRW}_\infty(C)$) one can argue, using Rayleigh’s monotonicity principle, that the limit on the r.h.s. of (2.2) exists by monotonicity (and the fact that probabilities are bounded) and is in fact independent of the choice of exhaustion. We conjecture that this also holds slightly less generally for finite β :

Conjecture 2.2. *For all $\beta > 0$ the limit $p_{\text{esc}}(a, \infty; V, C)$ exists for $\text{CTRW}_\beta(C)$ being the underlying random walk. Furthermore it is independent of the choice of exhaustion as long as the exhaustion converges to the limiting graph in the van Hove sense.*

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Numerics suggest that it may even exist independently of the exhaustion, but one has to be very careful here since the sequence of escape probabilities on the exhaustion may fluctuate (i.e. go up and down) quite a lot as we do not have a RMP for CTRW_β .

Note that for most purposes it is enough to bound the $\liminf$ of the escape probabilities, so this question does not seem to be crucial for applications.

Remark 2.3 (On the “naive” extension of electric network theory). *At this point one typically introduces objects like flows and relates them to the escape probability (via effective resistances). We will not do so since we could not find a natural generalisation of flows that is similarly useful for non-Markovian processes; note, however, that there is such a generalisation to non-reversible Markov chains [3] without totally asymmetric steps and it is noted that one might be able to extend results to the case with totally asymmetric steps. The $\text{CTRW}_\beta(C)$ can also be seen as some non-reversible Markov chain with totally asymmetric steps by remembering which subsets of $E \times [0, \beta)$ we already explored. This seems to be too complicated to keep track of in an efficient manner, however.*

Nonetheless we will borrow some language from the electric network language and refer to the origin a also as the *source* and call the point b to escape to the *sink*. As motivated by (2.2) one should think of the sink as boundary of some large box, but the following discussion will not make use of that.

2.3 Topological transformations to obtain bounds

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This toolbox of little tricks (of which we mention only two), or more precisely consequences of Rayleigh’s monotonicity principle, is also referred to as “Rayleigh’s short-cut methods” (see e.g. [6, p.73ff] for applications to proving recurrence/transience of the SRW on $\mathbb{Z}^d$ and other graphs).

2.3.1 Gluing vertices together

This is the limiting case as we send the conductance C_{xy} between two vertices x and y to infinity. In the simple random walk picture it is not hard to see that this effectively means that

we identify x and y , i.e. “glue them together”. In terms of crosses this means that the walker is “in a superposition of x and y with probability one half each” with no local time correlation, i.e. observing the walker being at x at time s does not provide any information whether the walker will be at x or y at time $s + ds$. It does however have strong periodic time correlations; think e.g. of the graph only consisting of x and y . Knowing the walker was at x at time s , we know that at time $s + \beta$ he can only be at y . This periodic time correlation does not allow us to interpret sending $C_{xy} \rightarrow \infty$ as identifying x and y if β is finite. This will become clearer in Subsection 3.1. Unfortunately this creates another problem to overcome since a naive extension of Rayleigh’s monotonicity principle will not cover the case of splitting vertices into two.

2.3.2 Splitting vertices

This is the limiting case as we send the conductance C_{xy} between two vertices x and y to zero. While it clearly still implies the same for both CTRW_β and the SRW, namely that the random walk cannot traverse the edge $\{x, y\}$ anymore, there is a subtle issue resulting from the previous problem of glueing vertices together: Since we cannot interpret a single vertex as two vertices connected by an edge with infinite conductance, the following “trick” will not work (naively, at least) for CTRW_β even if we managed to prove Rayleigh’s monotonicity principle.

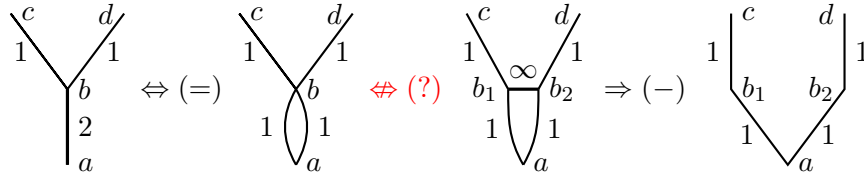


Figure 2: A sequence of topological transformations of a network. Even if we had “a” RMP for the CTRW , additional work would be required to prove that this does not increase escape probabilities since infinite conductance between two vertices cannot be naively interpreted as vertices that are glued together. (Signs indicate equality/decrease of escape probability.)

The conductances are written next to the edges. Note that the first step is actually an equivalence transformation (see Subsection 2.4), in the sense that it does not change the behaviour of CTRW_β at all. The last step (i.e. an inequality for escape probabilities) would be justified by Rayleigh’s monotonicity principle, but the second step fails since the CTRW_β on the two graphs behaves differently as described above.

2.4 Topological equivalence transformations

For electric networks several topological transformations are known to leave observables one typically cares about (such as escape probabilities) invariant. Here we analyse which ones carry over to our non-Markovian setting and which ones do not. The three most prominent ones are *conductances in parallel add*, *resistances in series add* and the *Y- ∇ transformation* (also known as Yang-Baxter equation or Delta-Star/Delta-Wye/T-II transformation and as a special case of the star-mesh transformation).

2.4.1 Conductances in parallel add

Let e_1, e_2 be two edges with the same endvertices and (potentially different) conductances c_1, c_2 , respectively. Then it is not hard to see that they may be replaced by a single edge with conductance $c_1 + c_2$. In fact this is even easier to see with the cross representation and using that the “sum” (i.e. union of crosses) of two Poisson point processes is another Poisson point process with intensity being the sum of intensities.

2.4.2 Resistances in series add

Define resistance $R = (R_e)_{e \in E} \in [0, \infty]^E$ as inverse of conductance, then we have for $e = \{x, y\}$ such that every path from x to y has to go through e :

$$p_{\text{esc}}(x, y) = \frac{C_{xy}}{C_a} \stackrel{(\text{def})}{=} \frac{1}{C_x R_{xy}},$$

with $\text{CTRW}_\beta(C)$ the underlying random walk for $\beta \in (0, \infty]$. For $\text{CTRW}_\infty(C)$ this holds more generally with x, y arbitrary and R_{xy} replaced by the so-called “effective resistance”, see e.g. [12]. For line segments this can be given explicitly as replacing two edges with resistances R_{xy}, R_{yz} with one edge with resistance $R'_{xz} = R_{xy} + R_{yz}$ does not change any electric properties of the network; in particular the escape probability is invariant under transformations such as:

$$\begin{array}{c} x \text{---} R_{xy} \text{---} y \text{---} R_{yz} \text{---} z \\ \circ \quad \quad \quad \circ \quad \quad \quad \circ \end{array} \Leftrightarrow (=) \begin{array}{c} x \text{---} R'_{xz} \text{---} z \\ \circ \quad \quad \quad \circ \end{array}$$

Note that, for unit conductances (i.e. $C_e = R_e = 1$), we have that a linear sequence of ℓ edges can be replaced by a single edge of conductance ℓ^{-1} ; in particular we merely have inverse linear decay of escape probabilities in (long) linear segments.

Compare this to the CTRW_β with $\beta < \infty$: Having a linear sequence of ℓ edges (henceforth referred to as linear segment of length ℓ) of unit conductance gives exponential (in ℓ) decay of the escape probability since every edge has a positive chance of not having any crosses at all!

Comparing the vast qualitative difference of decay of escape probabilities (ℓ^{-1} vs $O(\exp(-\ell))$), it does not seem plausible that a construction analogous to this adding of resistances can hold for the CTRW_β with $\beta < \infty$.

2.4.3 Star-mesh transform

Given how the previous section is a special case of the Y- ∇ transform (set conductance on one edge of the Y graph to 0) and hence more generally of the star-mesh transform, it does not seem plausible that the star-mesh transform has a natural extension to CTRW_β with $\beta < \infty$.

It is, however, instructive to state precisely and prove the star-mesh transformation as it provides insight why it (and hence in particular the resistances in series add principle) fails to hold for the CTRW . We follow the approach in [5], which might look unconventional for physicists interested in and used to work with (effective) resistances, but it can easily be seen to be equivalent.

First, we need to introduce some more notation: We consider a graph $G = (V, E)$ induced by conductances C and $G_x = (V_x, E_x)$, the reduced graph at x , with $V_x := V \setminus \{x\}$ and

$E_x := \{ \{y, z\} \in E : y, z \neq x \}$ and conductances

$$\tilde{C}_{yz} := C_{yz} + C_{yz}^{*,x} \quad \text{where} \quad C_{yz}^{*,x} := \frac{C_{xy}C_{xz}}{C_x}. \quad (2.3) \quad \{\text{eq:reduc}$$

It turns out to be convenient to introduce L , the infinitesimal generator of the random walk with conductances C , i.e.

$$-Lg(z) = \sum_{y \in V} C_{zy}[g(z) - g(y)] = C_z \sum_{y \in V} P_{zy}[g(z) - g(y)] = (D - A)g(z),$$

where $D := \text{diag}(C_x)_{x \in V}$ and $A_{xy} := C_{xy}$ are generalised degree and adjacency matrices, respectively. $-L$ (which is positive semi-definite) may thus be thought of as the (unnormalised) Laplacian on the weighted (by C) graph $G = (V, E)$.^{*} Similarly we define L_x as the infinitesimal generator of the random walk on G_x induced by conductances $\tilde{C}$ as defined in (2.3), i.e. for $g : V_x \rightarrow \mathbb{R}$ we have

$$-L_x g(z) = \sum_{y \in V_x} \tilde{C}_{zy}[g(z) - g(y)] = \tilde{C}_z \sum_{y \in V_x} \tilde{P}_{zy}[g(z) - g(y)],$$

where we implicitly defined the transition probabilities $(\tilde{P}_{zy})_{z,y \in V_x}$ of a new random walk on G_x .

Recall that we call a function $g : V \rightarrow \mathbb{R}$ harmonic at $z \in V$ w.r.t. $(P_{xy})_{x,y \in V}$ if

$$g(z) = \sum_y \frac{C_{zy}g(y)}{C_z} \stackrel{(\text{def})}{=} \sum_y P_{zy}g(y) \quad \Leftrightarrow \quad Lg(z) = 0.$$

Lemma 2.4 (Star-mesh transform). *Assume $g : V \rightarrow \mathbb{R}$ is P -harmonic at x . Then*

$$Lg(z) = \begin{cases} L_x g(z) & \text{if } z \in V_x \\ 0 & \text{if } z = x. \end{cases} \quad (2.4)$$

In particular we have that if $g : V \rightarrow \mathbb{R}$ is P -harmonic, then g restricted to V_x is $\tilde{P}$ -harmonic. For the SRW a lot of interesting quantities are P -harmonic, see the last bit of the proof for the SRW in Appendix A. One of them is the escape probability

$$g(z) := \mathbb{P}(\text{SRW started at } z \text{ hits } b \text{ before } a),$$

which is P -harmonic at all points in $V \setminus \{a, b\}$. Say we want to evaluate g at some points $w \in W \subseteq V \setminus \{a, b, x\}$. Now Lemma 2.4 states that it suffices to consider the random walk on G_x with transition probabilities $(\tilde{P}_{zy})_{z,y \in V_x}$ instead of the original one on the larger graph G . In particular we have that $p_{\text{esc}}(a, b; V, C)$ is invariant under star-mesh transformations removing vertices that are not adjacent to the source or the sink (at these points escape probabilities fail to be harmonic, so Lemma 2.4 does not apply).

^{*}With a sign convention that is at odds with Wikipedia, https://en.wikipedia.org/wiki/Laplacian_matrix

Proof of Lemma 2.4. We have

$$\begin{aligned} Lg(z) &= \left(\sum_{y \in V_x} C_{zy}[g(y) - g(z)] + C_{zx}[g(x) - g(z)] \right) \\ &= \sum_{y \in V_x} (C_{zy} + C_{zy}^{*,x})[g(y) - g(z)], \end{aligned}$$

where the last step follows from using the assumption that $g : V \rightarrow \mathbb{R}$ is P -harmonic at x , i.e. $g(x) = \sum_{y \in V} P_{xy}g(y)$, to write the second occurrence of $g(x) - g(z)$ as $\frac{1}{C_x} \sum_{y \in V_x} C_{xy}(g(y) - g(z))$. The claim follows from comparing with the definition of L_x and $\tilde{C}_{zy}$. $\square$

In [5] star-mesh transforms were used recursively in combination with the so-called octopus inequality, stating that the spectral gap of the interchange process does not decrease under star-mesh transforms, and noting that the spectral gaps of the interchange process and the SRW on maximally reduced graphs coincide, to prove that the spectral gaps of the interchange process and the SRW coincide (the other inequality is trivial since the spectrum of the SRW is included in that of the interchange process).

Remark 2.5. *In the physics/electrical network literature the following “terminology” seems to be more common:*

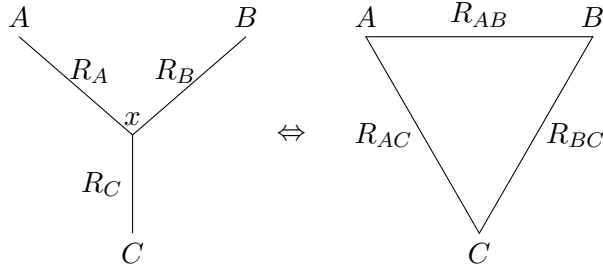


Figure 3: An (eponymous) illustration of the Y- ∇ transformation. Here the *resistances*, not the conductances, are written next to the edges. In the language above: $V = \{x, A, B, C\}$, $V_x = \{A, B, C\}$.

Here the resistances are related as follows:

$$\begin{aligned} R_{AB} &= \frac{R_A R_B + R_A R_C + R_B R_C}{R_C} & R_A &= \frac{R_{AB} R_{AC}}{R_{AB} + R_{BC} + R_{AC}} \\ R_{BC} &= \frac{R_A R_B + R_A R_C + R_B R_C}{R_A} & R_B &= \frac{R_{AB} R_{BC}}{R_{AB} + R_{BC} + R_{AC}} \\ R_{AC} &= \frac{R_A R_B + R_A R_C + R_B R_C}{R_B} & R_C &= \frac{R_{AC} R_{BC}}{R_{AB} + R_{BC} + R_{AC}} \end{aligned}$$

Remark 2.6. *The reason why the star-mesh transform (and as a special case the resistances in series add principle) does not generalise to the CTRW is now obvious: For finite β transition probabilities P_{zy} are not well-defined for the $CTRW_\beta$ as they depend strongly on the trajectory up to the current time.*

One might be tempted to start looking for alternatives by considering $\tilde{C}_{zy}$ s that may depend on the CTRW’s trajectory, but this is unlikely to lead to a solution that can be iterated multiple times to recursively reduce a complicated network.

Unfortunately, even bounds seem hard to obtain since the transition probabilities seem to be able to change a lot, both increasing and decreasing, depending on the past trajectory.

3 Towards a better understanding of a RMP for CTRWs

3.1 The counterexample

Consider the graph $G = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{b, c\}, \{b, d\}\}$ with source a and sink d :

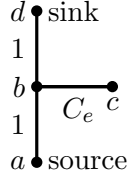


Figure 4: The counterexample where increasing C_e (the conductance of an edge not adjacent to the origin) *decreases* the escape probability. The conductances are written next to the edges.

The variable of interest, in particular as a function of C_e , is $p_{\text{esc}}(a, d; V, C) = p_{\text{esc}}(C_e)$. If RMP would hold in its full form we would have that increasing C_e would not decrease $p_{\text{esc}}(C_e)$.^{*} We note here that we can in fact observe a different behaviour (i.e. as C_e increases, $p_{\text{esc}}(C_e)$ decreases) in two regimes:

$0 < \beta \ll 1$: We consider the natural coupling[†] between the two instances X, X' of the CTRW_β on V with $C_e = 0$ and $C_e = 1$, respectively, started at a and run until they hit either a or d . Since almost surely there are only finitely many crosses on each edge, the points they “end up at” ($\in \{a, d\}$) are well-defined. Note that unless there is at least one cross on $\{a, b\}$ and $\{b, d\}$ both walks end up at a . Similarly, if there are less than two crosses on $e = \{b, c\}$ the points X and X' end up at will coincide again. Conditioning on the total number of crosses it is not hard to see that the dominant (for β sufficiently small) difference of the two escape probabilities comes from those configurations with four crosses in total: One on $\{a, b\}$, one on $\{b, d\}$ and two on $\{b, c\}$. The other configurations contribute by at least one order of magnitude in β less and may hence, for purposes of determining the sign of the difference of the escape probabilities, be ignored. In this case it is not hard to see that this can only make returning to a strictly more likely for X' . Thus the claim is proved for β sufficiently small.

$C_e \gg 1$: The above counterexample is not quite satisfactory as we are not interested in the $0 < \beta \ll 1$ regime — this is where percolation bounds are sharp. So consider $C_e \gg 1$ and X, X' as

^{*}For the SRW the harmonic function $h : v \mapsto \mathbb{P}(\text{SRW}_v \text{ hits } d \text{ before } a)$ for $v \in \{a, b, c, d\}$ can easily be computed explicitly and since c only has one neighbour and h is harmonic, their values have to coincide, i.e. $h(b) = h(c)$. Thus the escape probability for the SRW can be seen to be invariant under changes of C_e .

[†]I.e. sample X' as CTRW_β on V with $C \equiv 1$ for all edges and construct X by throwing away the crosses on $\{b, c\}$.

above. For the SRW ($\beta = \infty$) this would mean that b and c are glued together and thus we should expect no difference between X and X' . This counterexample is interesting for two reasons: On the one hand it shows that RMP does not hold (in its naive formulation) for *any* finite β , on the other hand it shows that we cannot interpret sending $C_e \rightarrow \infty$ as gluing together the two endpoints of the edge e as in the SRW.

Proof: Condition on the event that there is at least one cross on $\{a, b\}$ and the height of the first cross on $\{a, b\}$ is $h \in [0, \beta)$. For X we have the following schematic picture (note in particular that we cannot go up any further):

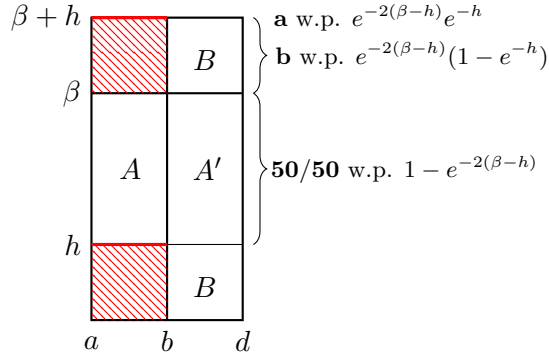


Figure 5: A schematic graphic of $E \times \beta$; dashed areas in red highlight that no crosses can be found here, since h is by assumption the height of the first cross on $\{a, b\}$. To the right are the probabilities of finding a cross in the respective areas with **a** indicating that if a cross is found in this area we end up at **a**; similarly for **b**. **50/50** indicates that if a cross is found in this area, we end up at a or b , both outcomes being equally likely.

We start at height 0 at a and travel to b at height h ; by assumption there are no crosses on $\{a, b\}$ below h , which is emphasised by the red background. By periodicity the same holds for the area between β and $\beta + h$. Note that the law of crosses in A and A' coincide: By definition the law of crosses in A' is that of a Poisson point process on (h, β) of intensity $C_{\{b, d\}} = 1$, where that of crosses on A is that of an independent Poisson point process on (h, β) conditioned to have at least one cross and the first cross being at height h . But once we have that there is a cross at height h the condition that there is at least one cross is already fulfilled and the remaining crosses are distributed independently according to a Poisson point process of intensity $C_{\{a, b\}} = 1$ on (h, β) .

Note that if there is at least one cross on $\{a, b\}$ or $\{b, d\}$, then it is equally likely that the first such cross will lead X back to a or to b . Hence, with probability $\mathbb{P}(\exists \text{ cross in } A \cup A') = 1 - e^{-2(\beta-h)}$ there is a 50/50 chance of either escaping to b or returning to a .

Now assume there is no cross in $A \cup A'$; this happens with probability $e^{-2(\beta-h)}$. Then if there is a cross in B our walker escapes to B . Again, the distribution of crosses on B is the law of a Poisson point process of intensity $C_{\{b, d\}} = 1$ on $[\beta, \beta + h) \simeq [0, h)$, hence $\mathbb{P}(\exists \text{ a cross in } B) = 1 - e^{-h}$.

Assume there are no crosses in $A \cup A' \cup B$, then the walker returns to a and thus failed to

escape. Summarising we get that

$$p_{\text{esc}}(C_e = 0) = \mathbb{P}(\exists \text{ a cross on } \{a, b\}) \int_0^\beta d\mathbb{P}(\text{first cross on } \{a, b\} \text{ is at height } h) \\ \times \left[\frac{1}{2} \left(1 - e^{-2(\beta-h)} \right) + 1 \cdot \left(e^{-2(\beta-h)} (1 - e^{-h}) \right) \right].$$

It remains to show a similar formula for the case with $C_e = \infty$, comparing integrands and concluding by continuity for C_e sufficiently large. Let us first sketch the situation just like above (and note again that we cannot go up any further – think about it!):

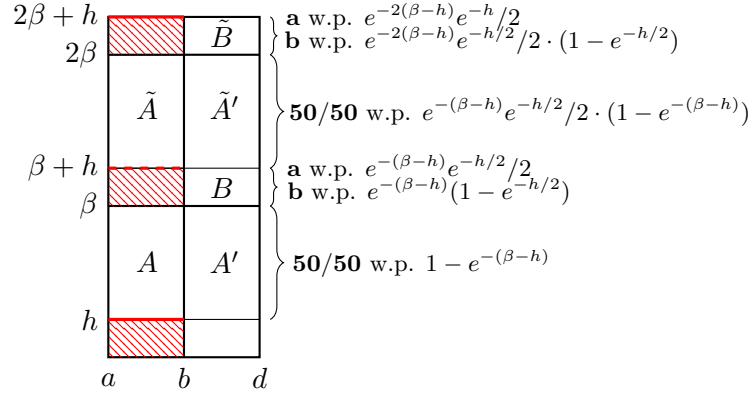


Figure 6: Same as last graphic, but this time there is a third edge that allows the walker to “overlook” crosses on the other two edges with probability $1/2$; this leads to different probabilities of finding crosses.

Again we start at a and go to b at height h . Now that we are at b , however, and there are infinitely many crosses to c , we will spend half of our time at c . Note, however, that since there are infinitely many crosses being at c at height s does not give you any information about whether you will still be at c at height $s+ds$. So effectively we might also ignore c in the graphic and just pretend to ignore every cross with probability $\frac{1}{2}$ until we reach height $\beta+h \equiv h$ again; if we reach it at all and did not go to a or d yet. In particular if we reach $\beta+h$, we must be on c since if we had been on b we’d have gone back to a . Hence we will now traverse all the crosses on $\{b, c\}$ in the reverse direction and discover all the crosses we “overlooked” (because we were on c at the same time mod β) before. Equivalently we may view $\tilde{A}, \tilde{A}', \tilde{B}$ as independent copies of A, A', B , respectively, where A, A', B are Poisson point processes of intensity $C_{\{a,b\}}/2, C_{\{b,d\}}/2$ and $C_{\{b,c\}}/2$, respectively. Also the cross on $\{a, b\}$ at height $\beta+h$ exists only with probability $\frac{1}{2}$, but the ones at heights h and $2\beta+h$ with probability 1. The corresponding probabilities to end up at a, d or a 50/50 chance

are obtained similarly to before. We thus end up with

$$p_{\text{esc}}(C_e = \infty) = \mathbb{P}(\exists \text{ a cross on } \{a, b\}) \int_0^\beta d\mathbb{P}(\text{first cross on } \{a, b\} \text{ is at height } h) \\ \times \left[\frac{1}{2} \cdot \left(1 - e^{-(\beta-h)} + e^{-(\beta-h)-h/2} / 2 \cdot (1 - e^{-(\beta-h)}) \right) \right. \\ \left. + 1 \cdot \left(e^{-(\beta-h)}(1 - e^{-h/2}) + e^{-2(\beta-h)-h} / 2(1 - e^{-h/2}) \right) \right].$$

We expect that for all $\beta > h > 0$ the integrand with $C_e = \infty$ should be dominated by the one with $C_e = 0$ so that $p_{\text{esc}}(C_e = \infty) < p_{\text{esc}}(C_e = 0)$. WolframAlpha*, however, suggests exactly the opposite!

Either way, we have a strict inequality which at least shows that sending $C_e \rightarrow \infty$ does *not* admit the interpretation of gluing the two endpoints of e together in terms of the CTRW_β ($\beta < \infty$).

3.2 Good and bad edges

One can check that for the first example we actually didn't need C_e to be exactly equal to 1. In fact any other value in $(0, \infty)$ would have worked as well (at the cost of possibly having to choose β even smaller). Now one may wonder how increasing C_e (while keeping all other $C_{e'}, e' \neq e$ fixed) affects the escape probability. If RMP would hold we would have that p_{esc} as a function of C_e is monotonically non-decreasing. As the above example showed, there can be edges e such that p_{esc} is actually decreasing in C_e though. Motivated by numerics we conjecture the following:

Conjecture 3.1 (Monotonicity of p_{esc}). *Fix a graph G with conductances C , equipped with some fixed source and sink. Fix any edge e not adjacent to the source. Now consider p_{esc} as a function of C_e , with the underlying random walk being the CTRW_β .*

Then p_{esc} is non-decreasing or non-increasing in C_e . In particular it is monotone in C_e .

The reason we exclude the case where e is adjacent to the source is in fact not the counterexample for RMP (Proposition 2.1), but the following:

$$\begin{array}{c} a & 0 & c & \text{---} & b \\ \circ & & \circ & & \circ \end{array} \Rightarrow (+) \quad \begin{array}{c} a & 1 & c & \text{---} & b \\ \circ & & \circ & & \circ \end{array} \Rightarrow (-) \quad \begin{array}{c} a & \infty & c & \text{---} & b \\ \circ & & \circ & & \circ \end{array}.$$

Here $e = \{a, c\}$ is adjacent to the source a and clearly increasing C_e from 0 to 1 can only increase the escape probability from a to b . But as we keep increasing C_e from 1 to ∞ the escape probability will go to 0 again since the probability of having two crosses on e before there is a single cross on $\{c, b\}$ goes to 1. This suggests that the “optimal” (in the sense that it maximises the escape probability) conductance on edges adjacent to the source is such that the probability of having exactly one cross before the first cross on one of its “good” (see upcoming definition) neighbours is maximised.

With Conjecture 3.1 it makes sense to define *good* and *bad* edges:

* [https://www.wolframalpha.com/input/?i=1%2F2*\(1-e%5E-\(x-y\)%2Be%5E-\(\(x-y\)%2By%2F2\)%2F2*\(1-e%5E-\(x-y\)\)\)+%2B*\(e%5E-\(x-y\)\(1-e%5E\(-y%2F2\)\)%2Be%5E-\(2\(x-y\)%2By\)%2F2*\(1-e%5E-\(y%2F2\)\)\)+%3E+1%2F2*\(1-e%5E-\(2\(x-y\)\)\)%2Be%5E-\(2\(x-y\)\)\(1-e%5E-y\)+and+x%3Ey%3E0](https://www.wolframalpha.com/input/?i=1%2F2*(1-e%5E-(x-y)%2Be%5E-((x-y)%2By%2F2)%2F2*(1-e%5E-(x-y)))+%2B*(e%5E-(x-y)(1-e%5E(-y%2F2))%2Be%5E-(2(x-y)%2By)%2F2*(1-e%5E-(y%2F2)))+%3E+1%2F2*(1-e%5E-(2(x-y)))%2Be%5E-(2(x-y))(1-e%5E-y)+and+x%3Ey%3E0)

Definition 3.2. Fix $\beta > 0$. For some fixed vertex set V we say an edge e is (C, β) -bad if p_{esc} is non-increasing in C_e . We say an edge e is (C, β) -good if p_{esc} is non-decreasing in C_e .

If additionally p_{esc} is not constant in C_e we call e strictly (C, β) -bad/good, respectively.

Whenever β is clear from context we will abuse notation and simply write C -good/bad; similarly for C .

3.2.1 Goodness of e does depend on $(C_{e'})_{e' \neq e}$

First of all, note that the above definition does not depend on C_e by Conjecture 3.1. It does, however, depend on the other $C_{e'}, e' \neq e$. Think e.g. of the two graphs in Figure 7* induced by the conductances C and C' , respectively (both conductances are assumed to be 1 wherever an edge is drawn and 0 otherwise, with the exception of $C'_f = \infty$):

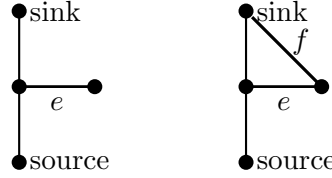


Figure 7: An example to illustrate that changing some $C_f, f \neq e$, may turn e from a bad edge (left) into a good edge (right).

Our counterexample, see Subsection 3.1, combined with Conjecture 3.1, shows that (at least for β sufficiently small) e in the l.h.s. of Figure 7 is a (C, β) -bad edge. However, it is not hard to see that e is a (C', β) -good edge since with probability 1 traversing e will result in the CTRW_β to hit the sink since we set $C'_f = \infty$.

3.2.2 Goodness of e does depend on β

Note that the dependence on β is not just a notational artefact, i.e. an edge e can be a strictly (C, β) -bad edge, but a strictly (C, β') -good edge as the following example[†] shows:

C_f is chosen very small, but positive – just how small exactly will be seen later in the proof. The shaded triangle represents a copy of an infinite 3-regular tree with unit conductances on each edge, rooted at the endpoint of e which is not shared with f . Additional to the endpoint of f which is not shared with e , we take points “at infinity”[‡] as sink. It is known that, for sufficiently small β , the CTRW_β on the infinite 3-regular tree is recurrent, i.e. has no chance of reaching the points at infinity, while there is a positive probability $\delta(\beta) > 0$ of escaping to infinity, i.e.

*One can construct an easier example by throwing away the lowest edge and setting the source to be the left endpoint of e ; we did not choose this example here as readers might be left to wonder whether it had something to do with e being adjacent to the source which is always tricky, even in the SRW picture.

[†]Again the counterexample would have worked if e, f were connected directly to the source, but we do not want to create the impression that this may be an artefact of e being adjacent to the source.

[‡]See (2.2) for an explicit construction.

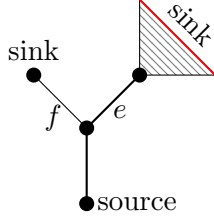


Figure 8: An example to illustrate that increasing β while keeping all conductances fixed may turn e from a bad edge (β small s.t. CTRW_β is not transient on $\mathbb{N}$) into a good edge (β large s.t. CTRW_β is transient on $\mathbb{N}$).

reaching the sink, for β sufficiently large. Now for β sufficiently small, s.t. the CTRW_β on the infinite 3-regular tree is recurrent, increasing C_e will decrease the escape probability (see e.g. our counterexample in Subsection 3.1). Fix some $\beta < \infty$ sufficiently large, i.e. such that $\delta(\beta) > 0$. For $C_e = 0$ the escape probability is clearly bounded from above by $1 - e^{-\beta C_f}$, the probability of f having at least one cross. We show that this upper bound is dominated by a lower bound for the escape probability once we set $C_e > 0$: For our fixed β we can choose C_f so small that this upper bound gets smaller than the probability of the CTRW_β traversing e before returning to the source (this is bigger than some constant depending on β and C_e) times the probability of traversing the 3-regular tree up to infinity before returning to e (which is bigger than $\delta(\beta) > 0$). This is a lower bound on the escape probability once $C_e > 0$ and thus we showed that increasing C_e from 0 to any positive value *increases* the escape probability iff β is sufficiently large. $\square$

Now it may be tempting to believe that increasing β can only make edges “less bad” (especially if one has in mind the $\beta \rightarrow \infty$ limit, i.e. the SRW, where no strictly bad edges exist). This, however, turns out to be wrong as well, as can be seen with a very similar example to the one above; only a sketch will be provided, the idea is again that for β sufficiently small, the only way to escape is via e , which makes e good, but for large β we can also escape via the tree and in fact this probability is larger if the escape probability via e is sufficiently small (just like before this is achieved by choosing C_f sufficiently small):

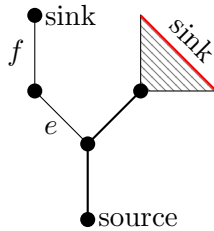


Figure 9: An example to illustrate that increasing β while keeping all conductances fixed may also have the opposite effect from before, i.e. turn e from a good edge (β small s.t. CTRW_β is not transient on $\mathbb{N}$) into a bad edge (β large s.t. CTRW_β is transient on $\mathbb{N}$).

3.2.3 Looking for “good” graphs

It is now tempting to look for graphs where *every* edge is good (in some appropriate sense) since for these graphs we would have monotonicity of the escape probability in β without having to quantify “how bad/good” edges are. As the above examples show (they can easily be extended to d -regular trees, $d \geq 3$) we should not even hope for d -regular trees ($d \geq 3$) to have only (C, β) -good edges for all pairs (C, β) , so some further restrictions are going to be needed.

First of all it is instructive to ask whether all edges not adjacent to the source* of $\mathbb{Z}$ (the 2-regular tree) are always good.

Conjecture 3.3. *Fix any $\beta > 0$ and any conductances C on $G = \mathbb{Z}$. Fix 0 as the source and $\{\pm n\}$, for some $n \in \mathbb{N}$ as the sink. Then every e not adjacent to the source is a (C, β) -good edge.*

Two more similar claims (in the same setting as in Conjecture 3.3) without the restriction that e must not be adjacent to the source would be:

- If $C \equiv 1$, then *every* edge e is (C, β) -good.
- If we restrict ourselves to $\mathbb{N}_0$ (e.g. by setting all the conductances of $\mathbb{Z}_-$ to zero), then every edge in $\mathbb{N}$ is (C, β) -good.

A more interesting question is about goodness of edges on d -regular trees where we do not allow arbitrary conductances $C \in \mathbb{R}_{>0}^E$ (we saw in Subsubsection 3.2.2 how this can go wrong), but restrict ourselves to “sufficiently interesting” ones:

Conjecture 3.4. *Fix any $\beta > 0$. Consider a rooted d -regular ($d \geq 2$) tree $G = (V, E)$ with the root being the source and as the sink the vertices at height n , for some $n \in \mathbb{N}$. Let $C \in \{1, 1+\varepsilon\}^E$ be so that once there is an edge e at distance m from the origin with conductance $C_e = 1$, then all edges at distance more than m must have conductance 1.*

Then there exists a (small) $\varepsilon = \varepsilon(\beta) > 0$ such that every edge e is (C, β) -good.

Ramifications of Conjecture 3.4: Effectively this allows us to repeatedly “increase all conductances at once by ε ” on d -regular trees ($d \geq 2$). In particular this implies monotonicity of p_{esc} in β and thus monotonicity of $\beta \mapsto \mathbf{1}\{\text{CTRW}_\beta \text{ is transient}\}$. The latter result is known for d -regular trees ($d \geq 3$) and was established by Hammond in [8, 9].

Why is Conjecture 3.4 formulated this way? The reason for choosing ε potentially very small is that there are competing effects, that we hope to be able to control when choosing ε sufficiently small. Consider for example Figure 10. The leftmost edge which is connected to the source may be bad if ε is too large since the escape probability on the left half of the tree is expected to be smaller than on the right half. On the other hand increasing its conductance should be good since the probability of there being no cross at all on any edges adjacent to the source decreases. For ε sufficiently small the latter effect should outweigh the former and hence make it a good edge.

*The reason for this restriction is that one can always choose the conductances to the, say, left of the source smaller than those to the right; then the first edge leading to the left would likely be bad.

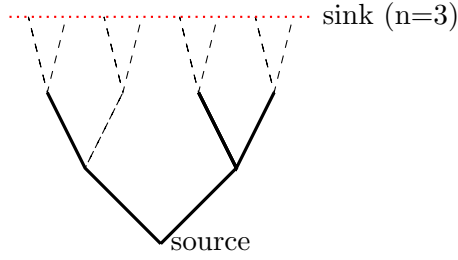


Figure 10: An illustration for a valid configuration of conductances on a 3-regular tree for Conjecture 3.4. Fat edges are edges with conductance $1 + \varepsilon$, dashed ones have conductance 1. Note that at height $m = 2$ there is an edge with conductance 1 and hence all edges at height $m > 2$ must have conductance 1 too.

{fig:good

3.3 Extending graphs

The motivation for trying to find an analogue of RMP was to get statements like “if G is transient then G' , obtained by adding good edges to G , is transient as well”. Since almost none of the tools from the theory of electric networks seem to carry over this seems to be out of reach for now, but we list some interesting conjectures supported by numerics in this subsection.

3.3.1 Slab monotonicity

Fixing two vertices a, b in a box $G_{\mathbf{L}} := \prod_{i=1}^d [0, L_i]^d \cap \mathbb{Z}^d$ where each edge has unit conductance one can add slabs to obtain $G'_L := [0, L_1 + 1] \times \prod_{i=2}^d [0, L_i]^d \cap \mathbb{Z}^d$. The escape probability from a to b will increase. Applying this procedure repeatedly (possibly rotated, so that not only the first coordinate is increased), we hope to obtain the following statements:

- The escape probability for the CTRW $_{\beta}$ on $\mathbb{Z}^d$ is well defined in a weak* sense. Compare with Conjecture 2.2.
- Starting with $G^n = \{0\} \times [-n, n]^{d-1}$, adding copies of itself should not decrease the escape probability. By the above argument we can also take $n \rightarrow \infty$. Hence the escape probability in $\mathbb{Z}^d$ (to the boundary of the n -ball) should be higher in higher dimensions. Using observation 5 this implies that $\beta_c(d) \geq \beta_c(d')$ whenever $d < d'$.

3.3.2 Transience for graphs with Cheeger constant ≥ 1

{subsubse

We now turn our attention to graphs where an analogue of RMP would actually imply transience of “bigger” graphs. More precisely we want to start with a d -regular tree ($d \geq 3$ so that it is transient for finite $\beta > \beta_c$) and “mollify” it using the steps sketched in Figure 11. After iteratively applying this procedure multiple times we should end up with a graph G' with Cheeger constant at least 1 since this is the Cheeger constant of the 3-regular tree which can, by construction, be isometrically embedded in this new graph. The fact that a 3-regular tree can be embedded *isometrically* in G' is quite important for the CTRW. While the SRW is transient on trees that

*As opposed to existence for all sequences of graphs converging to $\mathbb{Z}^d$ in the van Hove sense.

are obtained by inserting linear segments of length λ^k in the k -th generation of a d -regular tree (for $\lambda < d$, see Appendix B for a precise statement and a proof), the CTRW_β ($\beta < \infty$) is unlikely to be transient on such trees whenever $\lambda > 1$ due to its exponential decay of escape probability on linear segments (as opposed to the linear one for the SRW; see Subsubsection 2.4.2). This suggests also that connecting previously unconnected vertices has to increase escape probabilities significantly — otherwise we couldn't go from a graph that is recurrent for all $\beta > 0$, e.g. these trees with exponentially growing linear segments, to graphs like $\mathbb{Z}^3$ which are expected to be transient for β sufficiently large. It seems to be necessary to quantify (or at least bound from below) how much the escape probability is increased if we want to prove transience for CTRW_β on amenable graphs, or more generally: graphs with Cheeger constant < 1 .

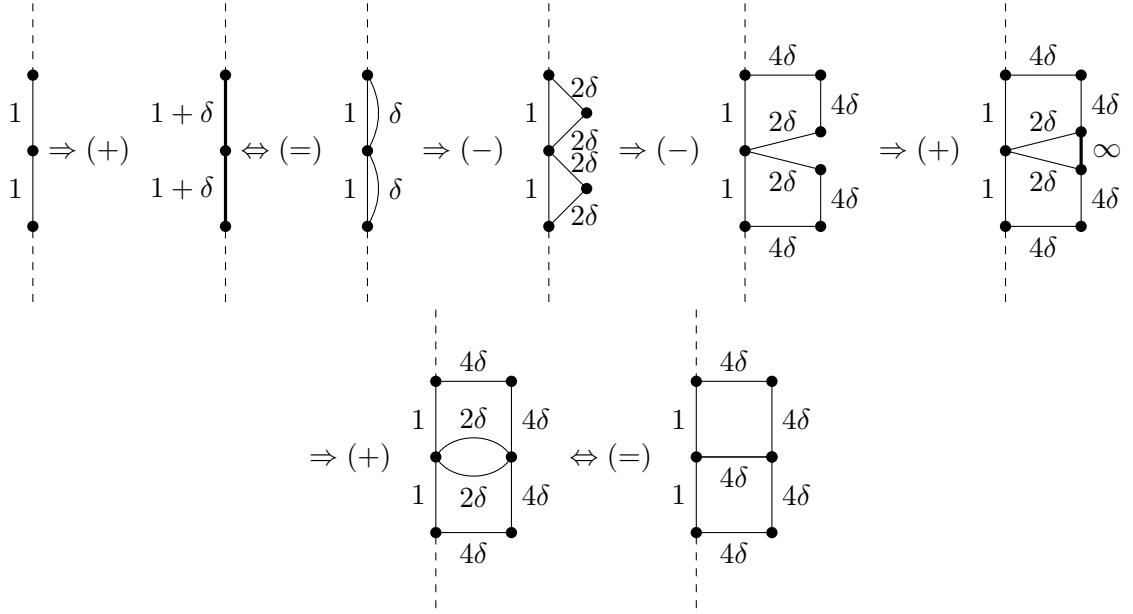


Figure 11: A sequence of topological transformations of the network that we conjecture to have net positive effect on (i.e. increase) the escape probability from one dashed end to the other.

{fig:moll

In Figure 11 the +’s (and -’s) indicate that we expect the escape probability to be increased (or decreased) after that step. The reasons are as follows:

- In the first step we increase conductances; since this (sub)graph is linear, in particular a tree (locally), we expect under suitably nice assumptions on the conductances and on β that increasing conductances by δ should not decrease the escape probability.
- In the second step we apply a topological equivalence transformation (see Subsubsection 2.4.1). This leaves the escape probability (and in fact virtually every other property of the network) unchanged.
- In the third and fourth step we insert vertices; for the SRW this is another topological equivalence transformation if the new conductances are chosen such that their inverses sum to the inverse of the original conductance (see Subsubsection 2.4.2), but for the CTRW_β

(with β finite) a percolation argument (outlined in Subsubsection 2.4.2) shows that this is not an equivalence transform since the escape probability decreases exponentially (CTRW) instead of linearly (SRW) in ℓ , the length of the linear segment. Due to the exponential decay and the fact that it leaves the escape probability invariant under the SRW (where we have linear decay) we believe that this transformation should decrease the escape probability.

- In the fifth step we add a new edge with infinite conductance. Since this does not connect parts of the graph to some kind of leaf or other structure from where it is unlikely to reach the boundary, we expect this to increase the escape probability.
- In the sixth step we glue the vertices together. For the SRW this would be a topological equivalence transformation, but for the CTRW we saw in our counterexample (see Subsection 3.1) that sending $C_e \rightarrow \infty$ is not equivalent to gluing the endvertices of e together and may in fact decrease the escape probability (i.e. having vertices glued together gives a higher escape probability than having an edge of infinite conductance between them).
- In the last step we simply repeat the argument from the second step.

Clearly we would have to be able to quantify the effect of these steps in order to conclude that the net effect is positive (so that transience of the original graph implies transience of its mollification). To this end it would be very helpful to have a lemma that allows to translate these local changes to some global change of the escape probability:

3.4 Translating local to global changes

As usual we start by asking whether this has been achieved for the SRW and the answer is (more or less) positive: Recall that the escape probability is (up to the factor C_a , the sum of conductances of edges adjacent to the source a) the inverse of the effective resistance. ([12, Proposition 2.5]) Note furthermore that the effective resistance (as a function of $V \times V \rightarrow [0, \infty]$) is a *metric* on the graph. (For a proof see e.g. [12, Proposition 2.19].) Putting it all together we obtain *for the SRW*

$$\frac{1}{C_a p_{\text{esc}}(a, c)} \leq \frac{1}{C_a p_{\text{esc}}(a, b)} + \frac{1}{C_b p_{\text{esc}}(b, c)},$$

which should, in this context, be interpreted as follows: If there is a high probability of escaping from a to b and a high probability of escaping from b to c , then there is a high probability of escaping from a to c . Note that this is sharp only if every path from a to c has to go through b — think of $V = \{a, b, c\}$ with edges induced by $C_{a,b} = C_{b,c} = 1$ and 0 otherwise. For the SRW one might then try to make this bound sharp in a general setting by summing over b in some minimal vertex cut set; since a path escaping from a to b may cross more than one vertex in this set we overcount considerably. An inclusion-exclusion argument should then give the exact value.

Now one might wonder if the same holds if we choose CTRW_β with $\beta < \infty$ to be the underlying random walk. Numerics for the above example with $V = \{a, b, c\}$ and edges induced by $C_{a,b} = C_{b,c} = 1, 0$ otherwise suggest that this is not true. (At least if one naively takes the formula from the SRW without adjustments.)

A Probabilistic proof of RMP for SRWs

{sec:prob}

An elementary and more importantly *purely probabilistic* proof of Proposition 2.1, i.e. RMP for the SRW, can be found in [6, p.55ff]. Noting that for many steps properties of the SRW are convenient, but strictly speaking not necessary, we present here a more abstract version in the hope of being able to generalise the result to a larger class of random walks that includes not only the SRW, but hopefully also the CTRW.

A.1 A generalised setup

Let us assume that $(X_t)_{t \in T}$ is a (potentially non-Markovian) random walk, i.e. a stochastic process with totally ordered index set T , law $\mathbb{P}$, and denote by $(\mathcal{F}_t)_{t \in T}$ its natural filtration, i.e. $\mathcal{F}_t = \sigma((X_s)_{s \leq t})$. For notational simplicity* we will assume that $T = \mathbb{R}_+$ and $X_t \in V \forall t \in \mathbb{R}_+$ for some countable V (think of it as vertices of some graph; strictly speaking, for the CTRW_β we would also have to consider the height, but as long as there are no double bars and only crosses, the height of X_t is deterministically given by $t \bmod \beta$). For any $\mathcal{F}_\infty$ -measurable random variable $F = f((X_t)_{t \in T})$ (e.g. $F = \mathbf{1}\{\tau_b < \tau_a^+\}$) we want to see how a change to the law of X affects $\mathbb{E}F$ (in our example: $\mathbb{E}F = \mathbb{P}(\tau_b < \tau_a^+) = p_{\text{esc}}$). To this end we introduce the martingale $(Y_t)_{t \in T}$ given by

$$Y_t := \mathbb{E}[F | \mathcal{F}_t],$$

which, in our example is just $\mathbb{P}(\tau_b < \tau_a^+ | (X_s)_{s \leq t})$, an $\mathcal{F}_t$ -measurable random variable. Note that we have

$$\mathbb{E}F \stackrel{(\text{def})}{=} \mathbb{E}Y_0 \stackrel{\text{mtg}}{=} \mathbb{E}Y_t \forall t = \mathbb{E}Y_\infty. \quad (\text{A.1}) \quad \{\text{eq:filtr}\}$$

Now consider a different measure $\mathbb{P}^{(\varepsilon)}$ for our random walk X with the same natural filtration (e.g. change the transition probability of one state to another by ε). The above question can now be rephrased as understanding the difference between $\mathbb{E}F$ and $\mathbb{E}^{(\varepsilon)}F$. We can “interpolate” between the two values of interest by noting that

$$Y_0 \stackrel{(\text{def})}{=} \mathbb{E}F (= \mathbb{E}^{(\varepsilon)}Y_0) \quad \text{and} \quad \mathbb{E}^{(\varepsilon)}Y_\infty = \mathbb{E}^{(\varepsilon)}F,$$

where the r.h.s. equality can be seen by noting that $Y_\infty = f((X_t)_{t \in T})$ and the distribution of $(X_t)_{t \in T}$ under $\mathbb{E}^{(\varepsilon)}$ is given by $\mathbb{P}^{(\varepsilon)}$.

We see that Y need not be a martingale with respect to $\mathbb{P}^{(\varepsilon)}$ since, in general, $\mathbb{E}^{(\varepsilon)}Y_0 = \mathbb{E}F \neq \mathbb{E}^{(\varepsilon)}F = \mathbb{E}^{(\varepsilon)}Y_\infty$. It is now instructive to understand where we pick up that difference, i.e. get

*Only for notational simplicity. We will not make use of these assumptions, but since we have a very concrete application in mind we do not want to be unnecessarily abstract.

a better understanding of the increments $\mathbb{E}^{(\varepsilon)}[Y_{t+dt} - Y_t]$. Plugging in definitions and using the shorthand $X_{[s,t]} := (X_r)_{r \in [s,t]}$ we get

$$\begin{aligned}
\mathbb{E}^{(\varepsilon)}[Y_{t+dt} - Y_t] &= \int d\mathbb{P}^{(\varepsilon)}(X) (\mathbb{E}[F|X_{[0,t+dt]}] - \mathbb{E}[F|X_{[0,t]}]) \\
&= \int d\mathbb{P}^{(\varepsilon)}(X_{[0,t]}) \left(\sum_x \mathbb{P}^{(\varepsilon)}(X_{t+dt} = x|X_{[0,t]}) \mathbb{E}[F|X_{[0,t]}, X_{t+dt} = x] - \right. \\
&\quad \left. \sum_x \mathbb{P}(X_{t+dt} = x|X_{[0,t]}) \mathbb{E}[F|X_{[0,t]}, X_{t+dt} = x] \right) \\
&= \sum_{s \in S} \int d\mathbb{P}^{(\varepsilon)}(X_{[0,t]}) \mathbf{1}\{X_t = s\} \\
&\quad \times \left(\sum_x \underbrace{(\mathbb{P}^{(\varepsilon)}(X_{t+dt} = x|X_{[0,t]}) - \mathbb{P}(X_{t+dt} = x|X_{[0,t]}))}_{=: \Delta p_{s \rightarrow x}^{(\varepsilon)}(X, t)} \underbrace{\mathbb{E}[F|X_{[0,t]}, X_{t+dt} = x]}_{=: v_x(X, t)} \right),
\end{aligned} \tag{A.2} \quad \{\text{eq:mtgIn}$$

where we introduced the set $S \subseteq V$ of points (say, minimal w.r.t. inclusion) such that there exists a trajectory* $X_{[0,t]}$ with $X_t \in S$ such that $\mathbb{P}^{(\varepsilon)}(X_{t+dt} = x|X_{[0,t]}) \neq \mathbb{P}(X_{t+dt} = x|X_{[0,t]})$. For example, if we change the conductance of some edge $e = \{r, s\}$, then $S = \{r, s\}$ because all the other transition probabilities are the same for both $\mathbb{P}$ and $\mathbb{P}^{(\varepsilon)}$. Furthermore we worked under the assumption that dt is chosen such that the random walk jumps at most once between t and $t + dt$.[†]

Intuitively (A.2) (the first line) means that we sample two instances of X with $X_{[0,t]}$ drawn from $\mathbb{P}^{(\varepsilon)}$ and $X_{[t+dt, \infty)}$ drawn from $\mathbb{P}$, but $X_{[t, t+dt]}$ drawn from $\mathbb{P}^{(\varepsilon)}$ for one instance and from $\mathbb{P}$ for the other instance of X . The expectations of F w.r.t. to these two random walks X are then subtracted.

A.2 Recovering the result for the SRW

We consider now the simple random walk with an added bridge of conductance ε between vertices p and q (i.e. $S = \{p, q\}$) and $F = \mathbf{1}\{\tau_b < \tau_a^+\}$. Since escape probabilities are invariant under time-reparameterisation we may assume without loss of generality that X jumps deterministically at integer times $t \in \mathbb{N}$ and hence choose $dt = 1$.

It remains to evaluate (A.2). Before calculating $\Delta p_{s \rightarrow x}^{(\varepsilon)}(X, t)$ and $v_x(X, t)$, note that neither of them depend on X or t since the simple random walk is a *time-homogeneous* (transition probabilities/expectations over future trajectories don't change in time t) *Markov chain* (transition probabilities/expectations over future trajectories don't depend on past trajectories).

*Strictly speaking a set of trajectories of non-vanishing measure under both $\mathbb{P}, \mathbb{P}^{(\varepsilon)}$ such that the following holds.

[†]To satisfy this we would have to choose dt as some random variable. In what follows it will sometimes be more convenient, however, to take $dt = 1/n$ for some large n that will be sent to ∞ . In our case (conductances are finite and graphs of bounded degree) it is not hard, although rather tedious, to make this rigorous as well.

- Evaluating $\Delta p_{s \rightarrow x}^{(\varepsilon)}(X, t)$:

$$\begin{aligned} \Delta p_{s \rightarrow x}^{(\varepsilon)} &\stackrel{(\text{def})}{=} \mathbb{P}^{(\varepsilon)}(X_{t+1} = x | X_t = s) - \mathbb{P}(X_{t+1} = x | X_t = s) \\ &= \mathbf{1}\{x = s'\} \frac{\varepsilon}{C_s + \varepsilon} + \frac{C_{sx}}{C_s + \varepsilon} - \frac{C_{sx}}{C_s} \\ &= \mathbf{1}\{x = s'\} \frac{\varepsilon}{C_s + \varepsilon} - \frac{\varepsilon}{C_s + \varepsilon} \frac{C_{sx}}{C_s} \end{aligned}$$

where s' is the other point (i.e. $s' \neq s$) in S that is connected to s via the new bridge of intensity ε .

- Evaluating $v_x(X, t)$: This is just the probability of the SRW (under $\mathbb{P}$) to escape to b before returning to a . In particular it is a harmonic function in x w.r.t. the transition kernel $\mathbb{P}$.

Multiplying these two terms and summing over x hence yields

$$\sum_x \Delta p_{s \rightarrow x}^{(\varepsilon)} v_x = \frac{\varepsilon}{C_s + \varepsilon} (v_{s'} - v_s).$$

Multiplying this by $\mathbf{1}\{X_t = s\}$, summing over $s \in S = \{p, q\}$ and taking expectation w.r.t. $\mathbb{E}^{(\varepsilon)}$ yields

$$\begin{aligned} \sum_t (\text{A.2}) &= \varepsilon \sum_t \mathbb{P}^{(\varepsilon)}(X_t = p) \frac{v_q - v_p}{C_p + \varepsilon} + \varepsilon \sum_t \mathbb{P}^{(\varepsilon)}(X_t = q) \frac{v_p - v_q}{C_q + \varepsilon} \\ &= -\varepsilon (v_p - v_q) \left(\frac{u_p^{(\varepsilon)}}{C_p + \varepsilon} - \frac{u_q^{(\varepsilon)}}{C_q + \varepsilon} \right), \end{aligned}$$

where we introduced $u_x^{(\varepsilon)} = \sum_t \mathbb{P}^{(\varepsilon)}(X_t = x) = \mathbb{E}^{(\varepsilon)}[\sum_t \mathbf{1}\{X_t = x\}]$ the expected number of times the SRW with the bridge of intensity ε on $\{p, q\}$ leaves the vertex x before reaching the sink or returning to the source.

By using* $u_x^{(\varepsilon)} \approx u_x$ (the same quantity but for the SRW without the bridge) for small ε and the fact that u_x/C_x is harmonic (and hence, up to an affine transformation that is determined by the boundary conditions[†], equal to v_x) we see that $v_r - v_s$ and $u_r^{(\varepsilon)}/(C_r + \varepsilon) - u_s^{(\varepsilon)}/(C_s + \varepsilon)$ must have the opposite sign for ε sufficiently small. Proposition 2.1 follows by noting that

$$\underbrace{\sum_t \mathbb{E}^{(\varepsilon)}[Y_{t+1} - Y_t]}_{\geq 0} = \mathbb{E}^{(\varepsilon)}[Y_\infty - Y_0] = p_{\text{esc}}^{(\varepsilon)} - p_{\text{esc}}. \quad \square$$

*A more detailed exposition of these last few steps can be found in [6, p.55ff]; we will not spend more time and space on this matter since it is a calculation that makes heavy use of the properties of the SRW and is thus unlikely to carry over to the setting we are interested in.

[†]In this case we have $u_a/C_a = 1/C_a$, $u_b/C_b = 0$, $v_a = 0$, $v_b = 1$; hence $u_x/C_x = (1 - v_x)/C_a$. (The C_a is not a typo!) Note that, unfortunately, our v is $1/C_a$ minus the v from [6].

A.3 Where are we stuck with the CTRW?

Our starting point will be (A.2) again, with the same F , but the underlying random walk being CTRW $_\beta$, i.e. $\mathbb{P}$ and $\mathbb{P}^{(\varepsilon)}$ denote different measures. Since the CTRW is not Markovian, one should be prepared for a much more complicated analysis. In particular $v_x(X, t)$ will indeed depend non-trivially on the entire trajectory.

Note that in the SRW picture before the observable F was invariant under time reparameterisation so we could choose $dt = 1$ after rescaling the SRW in such a way that it jumps deterministically at integer times t . In the CTRW picture we cannot do this anymore, since the times between jumps affect the future behaviour of the CTRW. It is hence more natural, albeit rather unpleasant to look at, to introduce the time discretisation into steps of size $1/n$. We will then jump at each fixed time t with some probability of order $\Theta(n^{-1})$. In what follows we will ignore the possibility of two (or more) crosses spawning at the same height since this is an event with probability no more than $O(n^{-2})$. Since our integrand, a difference of probabilities, is clearly bounded, we can safely disregard the contribution of such configurations in the limit $n \rightarrow \infty$. The discretisation will involve replacing $X_{[0,t]}$ by $(X_0, X_{\frac{1}{n}}, \dots, X_t)$, $\mathbb{P}$ and $\mathbb{P}^{(\varepsilon)}$ by $\mathbb{P}^{(n)}$ and $\mathbb{P}^{(\varepsilon, n)}$ respectively, with $\mathbb{P}^{(n)}$ and $\mathbb{P}^{(\varepsilon, n)}$ defined in the natural way.*

Similarly to before, in particular for s' as before, we obtain (assuming that there is no cross adjacent to x at height $t \bmod \beta$, which we can always do since we are conditioning on an event of probability $1 - \Theta(n^{-2})$) that for $s \in S$ we have

$$\Delta p_{s \rightarrow x}^{(\varepsilon)}(X, t) = \frac{1}{n} \left(\mathbf{1}\{x = s'\} \frac{\varepsilon}{C_s + \varepsilon} - \frac{\varepsilon}{C_s + \varepsilon} \frac{C_{sx}}{C_s} \right).$$

The fact that $v_x(X, t)$ depends on X and t prohibit us from getting a nice result as in the case of the SRW.

Remark A.1. *It is tempting to see whether one can derive conditions for time-homogeneous Markov processes to exhibit some analogue of RMP. This is an open question posed in [3]. In that case v_x and $\Delta p_{s \rightarrow x}^{(\varepsilon)}$ are independent of X, t , but determining the sign of (A.2) is still going to be quite hard, but might be worth looking into more.*

We do get, however, the following:

$$\begin{aligned} \sum_x \Delta p_{s \rightarrow x}^{(\varepsilon)}(X, t) v_{s \rightarrow x}(X, t) &= \sum_x \frac{1}{n} \left(\mathbf{1}\{x = s'\} \frac{\varepsilon}{C_s + \varepsilon} - \frac{\varepsilon}{C_s + \varepsilon} \frac{C_{sx}}{C_s} \right) v_{s \rightarrow x}(X, t) \\ &= \frac{\varepsilon}{n C_s + \varepsilon} \left[v_{s \rightarrow s'}(X, t) - \sum_x \frac{C_{sx}}{C_s} v_{s \rightarrow x}(X, t) \right] \\ &= \frac{\varepsilon}{n C_s + \varepsilon} \left[\mathbb{E}[F | X_0, \dots, X_{t-\frac{1}{n}}, s, s'] - \mathbb{E}[F | X_0, \dots, X_{t-\frac{1}{n}}, s] \right] \end{aligned}$$

where we wrote $v_{s \rightarrow x}(X, t)$ for $v_x(X, t)$ to emphasise that $X_t = s, X_{t+1/n} = x$ (if $X_t \neq s$ it will not contribute in (A.2)). The last equality can be seen by noting that $\frac{C_{sx}}{C_s}$ are just the transition

*There are of course some ambiguities when discretising time, but they can be resolved arbitrarily as their probability vanishes as $n \rightarrow \infty$. For the sake of a more accessible exposition we shall not be too rigorous here.

probabilities under $\mathbb{P}$ and the fact that the current expectation of any process is equal to the (adequately weighted) sum of future expectations.

When trying to repeat the remaining steps as for the SRW we notice the lack of symmetry:

$$\begin{aligned} \sum_t (\text{A.2}) &= \frac{\varepsilon}{n} \sum_{t \in \mathbb{N}/n} \left\{ \int d\mathbb{P}^{(\varepsilon)}(X_0, X_{\frac{1}{n}}, \dots, X_t) \mathbf{1}\{X_t = p\} \frac{\mathbb{E}[F|X_0, \dots, X_{t-\frac{1}{n}}, p, q] - \mathbb{E}[F|X_0, \dots, X_{t-\frac{1}{n}}, p]}{C_p + \varepsilon} \right. \\ &\quad \left. + \int d\mathbb{P}^{(\varepsilon)}(X_0, X_{\frac{1}{n}}, \dots, X_t) \mathbf{1}\{X_t = q\} \frac{\mathbb{E}[F|X_0, \dots, X_{t-\frac{1}{n}}, q, p] - \mathbb{E}[F|X_0, \dots, X_{t-\frac{1}{n}}, q]}{C_q + \varepsilon} \right\} \\ &\xrightarrow{n \rightarrow \infty} \varepsilon \int dt \left\{ \frac{\tilde{u}_p^{(\varepsilon)}}{C_p + \varepsilon} \int d\mathbb{P}_{t,p}^{(\varepsilon)}(X) (\mathbb{E}[F|X_{[0,t]}, X_{t+} = q] - \mathbb{E}[F|X_{[0,t]}]) \right. \\ &\quad \left. + \frac{\tilde{u}_q^{(\varepsilon)}}{C_q + \varepsilon} \int d\mathbb{P}_{t,q}^{(\varepsilon)}(X) (\mathbb{E}[F|X_{[0,t]}, X_{t+} = p] - \mathbb{E}[F|X_{[0,t]}]) \right\}, \end{aligned}$$

where, similarly as in the SRW case, we introduced the *dwelling times*

$$\tilde{u}_x^{(\varepsilon)} := \int \mathbb{P}^{(\varepsilon)}(X_t = x) dt = \mathbb{E}^{(\varepsilon)} \left[\int \mathbf{1}\{X_t = x\} dt \right],$$

the expected (under $\mathbb{P}^{(\varepsilon)}$) time X spends in total at site x and the conditioned measure $\mathbb{P}_{t,x}^{(\varepsilon)} = \mathbb{P}^{(\varepsilon)}(\cdot | X_t = x)$. The last step is just a Riemann integral with $dt = 1/n$.

It seems difficult to proceed any further from here without making any more assumptions. An obvious one (to recover as much as possible from the SRW setting) is to choose $\beta \gg 1$ and $0 < \varepsilon \ll 1$. In that case it seems natural to conjecture the following:

Conjecture A.2. *As $\beta \rightarrow \infty$ we have $\tilde{u}_x^{(\varepsilon)} = \tilde{u}_x^{(\varepsilon)}(\beta) \rightarrow \tilde{u}_x^{(\varepsilon)}(\infty) = u_x^{(\varepsilon)}$, the expected duration that a SRW with the same conductances spends on x before returning to its source or escaping to the sink.*

In particular, letting $\beta \rightarrow \infty$ and $\varepsilon \rightarrow 0$ we expect that

$$\frac{\tilde{u}_x^{(\varepsilon)}}{C_x + \varepsilon} \rightarrow \frac{\bar{v}_x}{C_a},$$

where $\bar{v}$ is the unique $\mathbb{P}$ -harmonic function with boundary conditions $\bar{v}_a = 1, \bar{v}_b = 0$ (i.e. the v in [6], but not our v from above).

It remains to understand $\int d\mathbb{P}_{t,p}^{(\varepsilon)}(X_{[0,t]}) (\mathbb{E}[F|X_{[0,t]}, X_{t+} = q] - \mathbb{E}[F|X_{[0,t]}])$ for each fixed t . Intuitively this means we first sample $X_{[0,t]}$ from $\mathbb{P}^{(\varepsilon)}$ conditioned to be on p at time t and then compare the probability of escaping to the sink (under the CTRW without the ε -bridge, i.e. $\mathbb{P}$) between $X_{[0,t]}$ and $X_{[0,t]}$ conditioned to make a jump to q at time t .

B A sufficient condition on certain tree-like subgraphs of $\mathbb{Z}^3$ to be transient (for the SRW)

Before we state and outline the proof of [7, Proposition 3], we want to motivate why the result should seem plausible: Assume you start with some rooted, d -regular tree T , i.e. every vertex

{sec:tree

has exactly d children. On the edges connecting the k -th with the $k + 1$ -th generation we put conductances λ^{-k} . Instead of considering this random walk directly, look instead at the evolution of its height. For this it only matters whether the walker chose to visit a child or its parent and clearly it is enough to see whether this process goes off to infinity before returning to zero to analyse transience of the original random walk on the tree. The probability of going up can now easily be seen to be $\frac{d\lambda^{-(k+1)}}{d\lambda^{-(k+1)} + \lambda^{-k}}$ which is greater than $\frac{1}{2}$ iff $d > \lambda$. In this case the process describing the height of the walker has a drift away from 0 (uniformly in k), hence there is a positive probability of it reaching ∞ before returning to its origin.

Remark B.1. *In terms of the usual random walk (that is identified with $CTRW_\infty(C)$) and $\lambda \in \mathbb{N}$ the above admits the equivalent (see Subsubsection 2.4.2) interpretation of considering a tree where vertices have exactly one child, unless they are at heights $(\lambda^k)_{k \in \mathbb{N}}$, in which case they have $d - 1$ children.*

*So the crucial insight here is that we can take a d -regular tree and insert linear components of length that may grow exponentially in the height k we are inserting them at (i.e. like λ^k with $\lambda < d$) and the random walk will still remain transient. This is crucial for purposes of showing transience in $\mathbb{Z}^3$, since if we want to embed an infinite d -regular tree into $\mathbb{Z}^3$ such that no two different vertices are mapped to the same point in $\mathbb{Z}^3$ and edges in the tree are mapped to edge- and site-disjoint paths in $\mathbb{Z}^3$, we have to have exponential growth of the lengths of these paths.**

Note that even if we could condition on there being at least one cross on each edge (this usually comes at the price of having to select the tree as a subgraph of the random infinite percolation cluster with $p = 1 - e^{-\beta}$, but can be done in $\mathbb{Z}^d$, $d \geq 3$, see e.g. [7]) this would not work either as there is still a positive probability of an edge being completely blocked, e.g. by a “double cross” as defined in [10].

For [7, Proposition 3] topological transformations as discussed in Subsections 2.3, 2.4 are used to extend this to “perturbations” of trees as described above.

*This can easily be seen by comparing the asymptotic surface growth of n -balls in the tree (exponential) and $\mathbb{Z}^d$ (polynomial).

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