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1 Introduction and Notation

1.1 The unweighted case

We fix a countable, locally finite graph $G = (V, E)$. To the unweighted random interchange process on G with $T = \text{Poisson}(\beta|E|)$ transpositions in total* (i.e. $\text{Poisson}(\beta)$ transpositions/crosses on each edge) one typically considers the following natural coupling to a Bernoulli edge percolation process on G with parameter $p = \mathbb{P}(\text{Poisson}(\beta) > 0) = 1 - e^{-\beta}$: Consider an edge open if there is at least one cross and closed otherwise. In the unweighted case the distributions of crosses on different edges are independent and thus the resulting percolation process really is a product measure which we denote by π_p .

1.2 The difficulty with the weighted case

If we consider the weighted interchange process (see e.g. [Uel17, Section 2.4], assume $\theta > 1$ so we want more—hence smaller—cycles, otherwise we can easily use

*Note that if we consider sequences of such graphs, say d -regular, where d may diverge, then we should replace β by β/d to stay in the interesting parameter regime. As a rule of thumb things become interesting when the associated CTRW makes a jump after $\text{Exp}(O(1))$ time, i.e. when there are $O(1/d)$ crosses per edge.

the product measure bound), however, then this introduces dependencies between configurations of crosses. In particular the above product measure bound will not hold.

1.3 What do we want and why do we want it?

One can still derive some rough bounds which will be outlined below, but ultimately one would like to have some bound in terms of the random cluster model which seems to be the natural analogue to the Bernoulli product measure above.

There are multiple reasons to be interested in such bounds: Firstly, they are the easiest way to establish that for low β no macroscopic cycles occur, which can be seen as a justification for why the problem of looking for macroscopic cycles for large, but finite β is not just a “trivial” matter. Secondly, the product measure bound is conjectured to be sharp for the unweighted interchange process on sequences of graphs of diverging vertex degree in the sense that the critical parameters coincide. [Bjö16] observed something similar for the weighted interchange process on the complete graph if compared to the random cluster model. This motivates finding a natural coupling between the weighted interchange process and the random cluster model that provides a rigorous (although not sharp in general) bound of the critical parameters, also on bounded degree graphs.

2 A rough product measure bound

The central idea is to note that the law of the weighted interchange process is defined as a Radon-Nikodym derivative with respect to the law of the (unweighted) interchange process and that this derivative ($\theta^{\#\text{cycles}}$) doesn’t distort the measure “too much” in the following sense:

Consider a fixed edge e . We want to obtain bounds for the probability of finding a certain number of crosses on e . Clearly every cross can introduce/remove at most one cycle and hence, if A_k denotes the event of e having exactly k crosses, then

$$\mathbb{P}_\beta(A_k)\theta^{-k} \leq \mathbb{P}_{\beta,\theta}(A_k|X) \leq \mathbb{P}_\beta(A_k)\theta^k,$$

where $\mathbb{P}_{\beta,\theta}$ is the law of the weighted random interchange (with parameter θ), X is any event “on configurations on edges $\tilde{E} \subseteq E : e \notin \tilde{E}$ ”, and $\mathbb{P}_\beta = \mathbb{P}_{\beta,1}$ is the law

of the unweighted random interchange process. In particular we have $\mathbb{P}_\beta(A_k|X) = \mathbb{P}_\beta(A_k) = \mathbb{P}(\text{Poisson}(\beta) = k) = e^{-\beta} \frac{\beta^k}{k!}$ and hence

$$e^{\beta(1/\theta-1)} \mathbb{P}(\text{Poisson}(\beta/\theta) = k) = e^{-\beta} \frac{(\beta/\theta)^k}{k!} \leq \mathbb{P}_{\beta,\theta}(A_k|X) \leq e^{-\beta} \frac{(\beta\theta)^k}{k!} = e^{\beta(\theta-1)} \mathbb{P}(\text{Poisson}(\beta\theta) = k).$$

In particular we have for $B := \bigcup_{k \geq 1} A_k$, the event that there is at least one cross on e :

$$\mathbb{P}_{\beta,\theta}(B|X) \leq e^{\beta(\theta-1)} (1 - e^{-\beta\theta}) =: p. \quad (2.1) \quad \{\text{eq:defp}\}$$

By [LSS⁺97, Lemma 1.1] we get the stochastic domination of $\mathbb{P}_\theta \circ \mathbf{1}_B$ by the Bernoulli product measure π_p . Note that we cannot have an infinite cycle unless there is an infinite cluster of the $\mathbf{1}_B$ process, which is dominated by π_p . Putting it all together we get that $\beta_c(\theta) := \inf\{\beta > 0 : \mathbb{P}_{\beta,\theta}(\exists \text{ an infinite cycle}) > 0\}$ is larger than $\beta_c^{\text{per}}(\theta) := \inf\{\beta > 0 : \pi_p(\exists \text{ an infinite cluster}) > 0\}$, where p is defined as in (2.1), which is perhaps better illustrated by Figure 1 (for fixed $\theta \geq 1$):

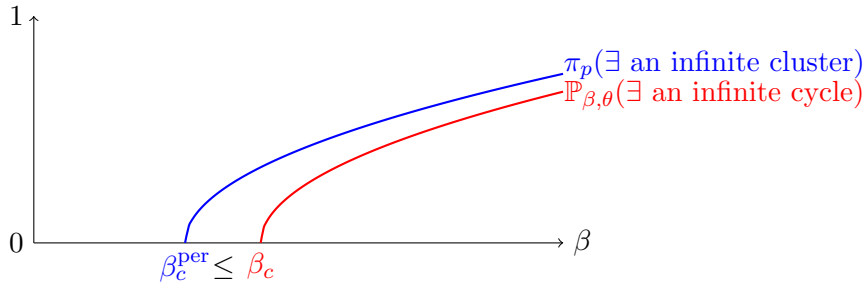


Figure 1: A schematic figure to illustrate how $\mathbb{P}_\theta \circ \mathbf{1}_B \preceq \pi_p$ implies $\beta_c^{\text{per}} \leq \beta_c$. (And nothing more; there are no claims about continuity or even monotonicity of the functions involved. We do know, however, that both functions are 0 for β sufficiently small.) \{\text{fig:critParams}\}

Note that usually it is easier to give bounds on the critical parameter for percolation on graphs, so it is of interest to get invert the map $\beta \mapsto e^{\beta(\theta-1)} (1 - e^{-\beta\theta})$; most importantly for $\theta = 2$ since this corresponds to the quantum Heisenberg ferromagnet. In that case we have $\beta(p) = \log \left(\frac{1}{2} \left(p + \sqrt{p^2 + 4} \right) \right)$, in particular we have that

$$\beta_c \geq \log \left(\frac{1}{2} \left(p_c + \sqrt{p_c^2 + 4} \right) \right),$$

where $p_c = p_c(G, \text{bond})$ is the critical bond percolation parameter for the graph G and for $\theta = 2$.

3 Bounds for more general quantum spin systems

Using [FU15, Theorem 3.2] we get that there is exponential decay of correlations (i.e. no macroscopic cycles in the loop representation of the QHF $\Leftrightarrow \theta = 2$) for $\beta < \frac{1}{2d} \sup_a a e^{-\frac{3}{2}a} = \frac{1}{3de}$ on $G = \mathbb{Z}^d$. In particular this gives that

$$\beta_c > \frac{1}{3de}.$$

Note that, on $\mathbb{Z}^2$ (where we know $p_c = 1/2$) this bound is weaker than the naive product measure domination bound, i.e. $\beta_c \geq \log \left(\frac{1}{2} \left(p_c + \sqrt{p_c^2 + 4} \right) \right) \approx 0.247 > \frac{1}{12e}$. Using the bound* $p_c(G^{(\Delta)}, \text{bond}) \geq \frac{1}{\Delta-1}$, where $G^{(\Delta)}$ is any connected graph with vertex degree bounded by Δ , we get for general $\mathbb{Z}^d$ (where $\Delta = 2d$) that $\beta_c \geq \log \left(\frac{1}{2} \left(\frac{1}{2d-1} + \sqrt{\frac{1}{(2d-1)^2} + 4} \right) \right)$. According to WolframAlpha this is bigger than $\frac{1}{3de}$ for all $d \geq 1$ and hence provides a better lower bound on β_c .

4 Attempting to derive a natural bound in terms of the random cluster model

Assume for now that $G = (V, E)$ is finite. Recall that the probability of finding a configuration $\sigma \in \{0, 1\}^E$ is given by

$$\mathbb{P}_{rc}(\sigma) \propto p^{o(\sigma)} (1-p)^{c(\sigma)} q^{K(\sigma)} = (1-p)^{c(\sigma)} \prod_{C \text{ cluster}(\sigma)} p^{|C|} q, \quad (4.1)$$

where $o(\sigma)$ ($c(\sigma)$) is the number of open (/closed) edges and $K(\sigma)$ is the number of open, connected clusters.

Now we try to write the probability of finding a configuration $\omega = (\omega_e)_{e \in E}$, where each ω_e is (a.s.) a finite sequence of real numbers (“heights of crosses on e ”) in $[0, \beta)$. We denote its length $|\omega_e|$ by n_e . Let $\omega_\downarrow$ be its projection on E , i.e. $(\omega_\downarrow)_e := \mathbf{1}\{n_e \geq 1\}$, the indicator whether there is at least one cross on e . Clearly each $\omega_\downarrow$ can be seen as some random cluster configuration σ and thus we will extend notions like $o(\cdot), c(\cdot)$ and clusters to $\omega_\downarrow$. Then we have

$$\mathbb{P}_{\beta, \theta}(\omega : \omega_\downarrow = \sigma) \propto (e^{-\beta})^{c(\sigma)} \prod_{C \text{ cluster}(\sigma)} (1 - e^{-\beta})^{|C|} \theta \times \text{correction}(C), \quad (4.2)$$

*This is the usual SAW bound.

where the correction is given by

$$\text{correction}(C) = \theta^{-1} \sum_{m=1}^{|V|_C|} \theta^m \mathbb{P}_{\beta, \theta}(M|_C(\omega) = m | (n_e)_{e \in E|_C} \geq 1),$$

where $V|_C$ and $E|_C$ are the vertices and the edges of the connected component C , respectively and $M|_C$ is the number of cycles of the permutation induced by ω restricted to C . E.g. if $n_e(\omega) = 1$ for all $e \in E|_C$ and C is a tree, then we know that $M|_C(\omega) = 1$ since no transpositions “cancel”.

4.1 On the correction

Ideally we would like the correction to be of order $O(1)$ (uniformly in all “likely” C) as the number of vertices goes to infinity, but that is unreasonable in general since this would imply that the critical parameters of the random cluster model and the QHF coincide (if their density is the same up to some non-zero factor of $O(1)$, then they are mutually absolutely continuous). So this may be expected for certain nice graphs of diverging vertex degree like the complete graph and maybe(?) the hypercube. In general, however, it seems reasonable that the dominant contribution in the sum in the definition of the correction term comes from $m \ll |V|_C|$. (This is because we expect that once all edges in some graph C support at least one transposition, then the probability that we get only microscopic cycles again should be very small; in particular it will hopefully decay faster than θ^m .)

4.2 On the complete graph

On the complete graph [Bjö16] proved that the critical parameters of the random cluster model with $q = \theta$ and $p = 1 - e^{-\beta}$ and the QHF coincide, this means that giant cycles and giant clusters should appear at the same time.

References

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