

Loop soups & percolation

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What are loop soups and why do we care?

“Def:” A loop soup is a probability measure on collections of closed trajectories.

Example: A probability measure on permutations.

Setup:

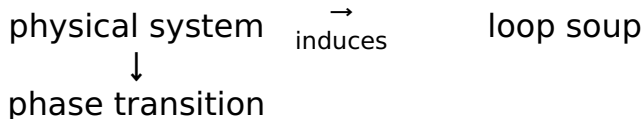
physical system
↓
phase transition

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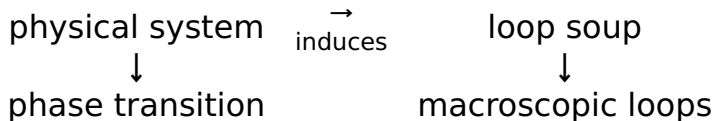


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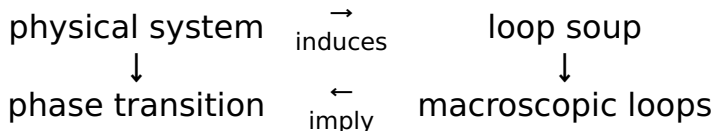


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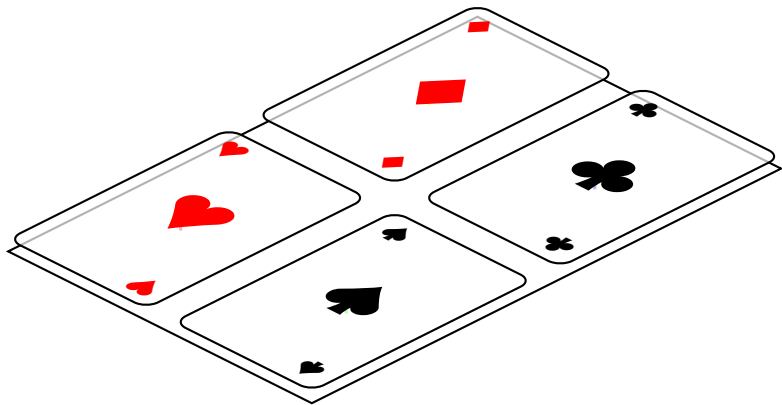
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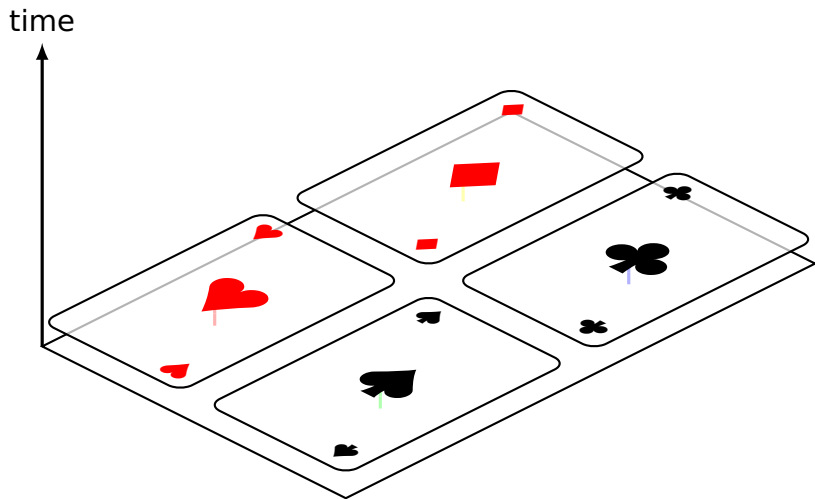
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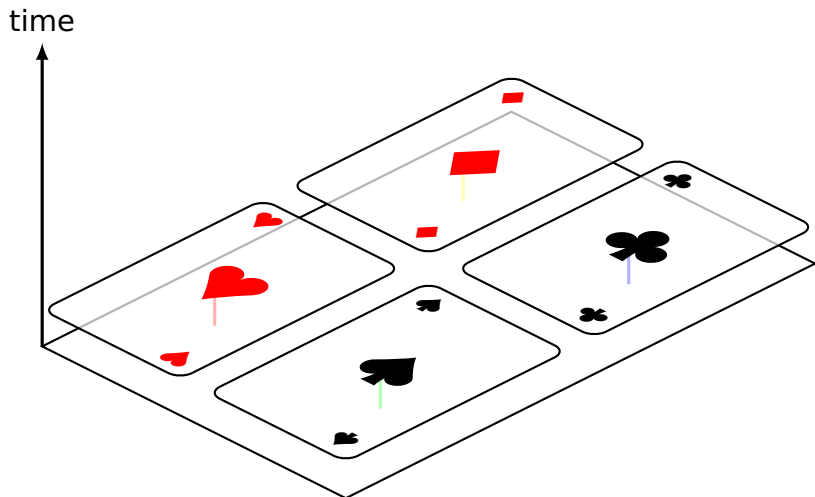
A concrete example: Shuffling adjacent cards



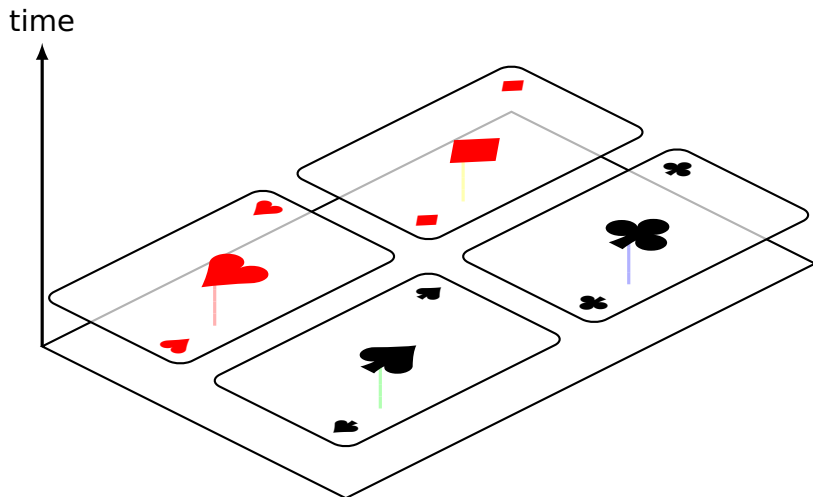
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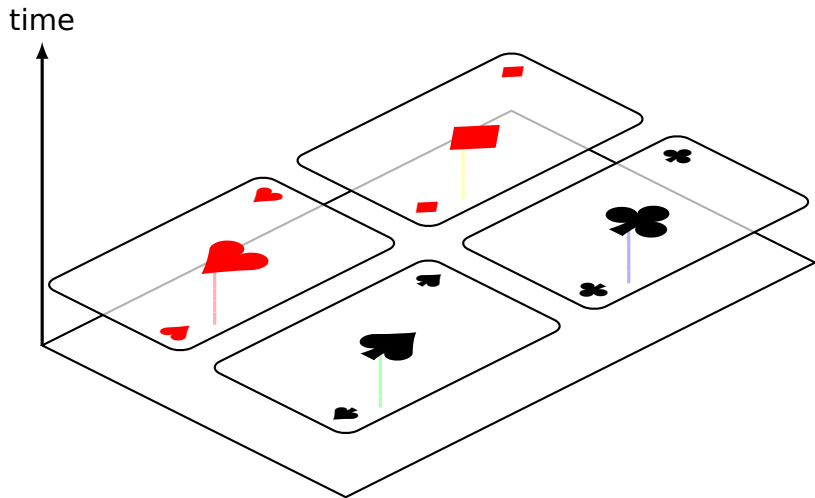
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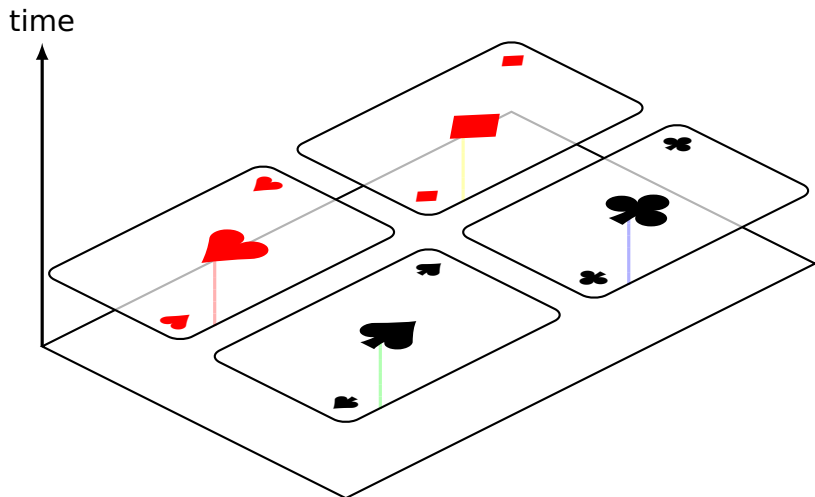
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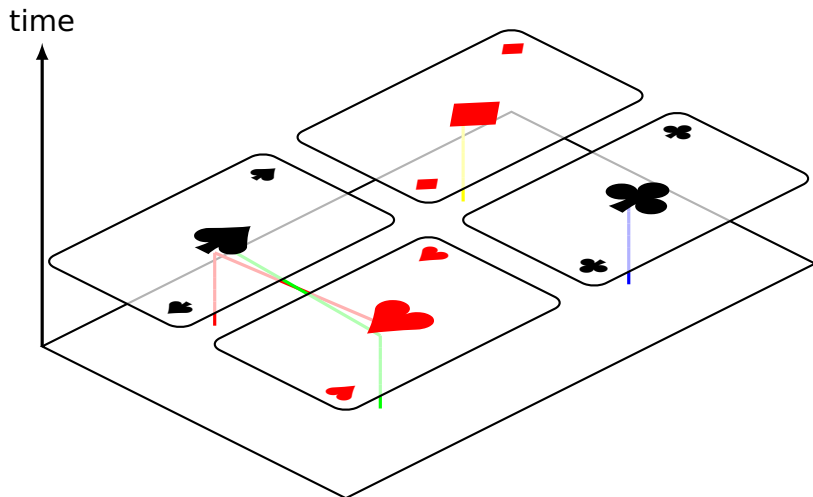
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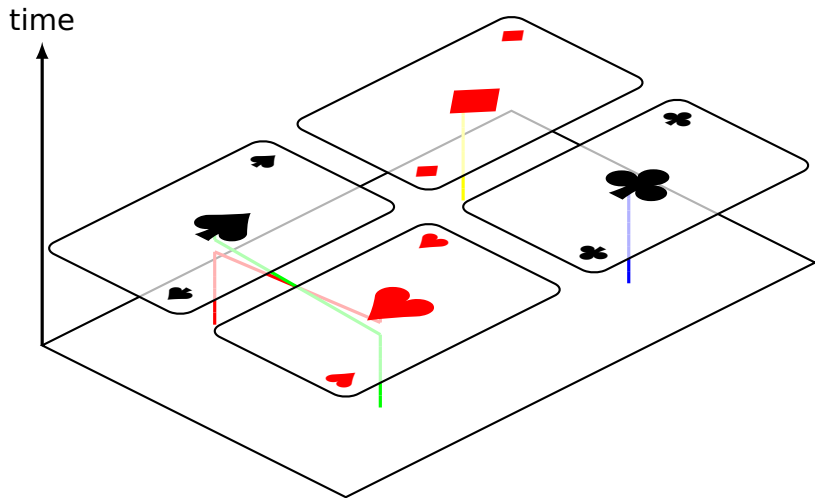
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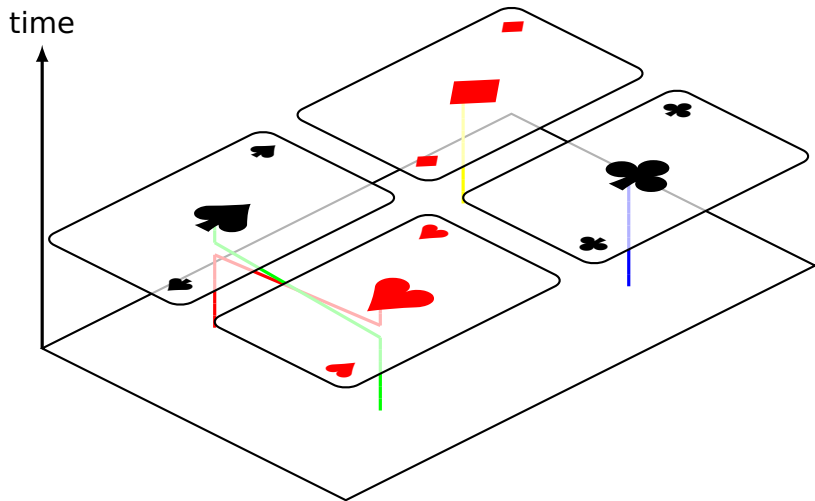
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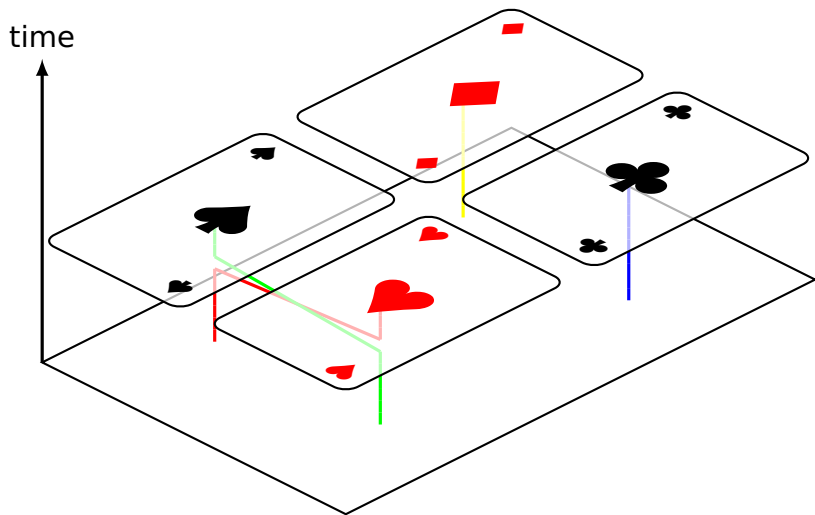
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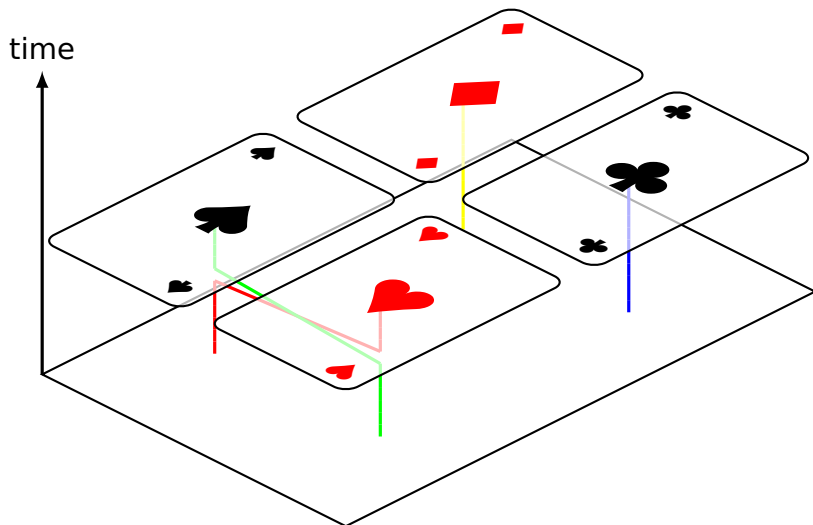
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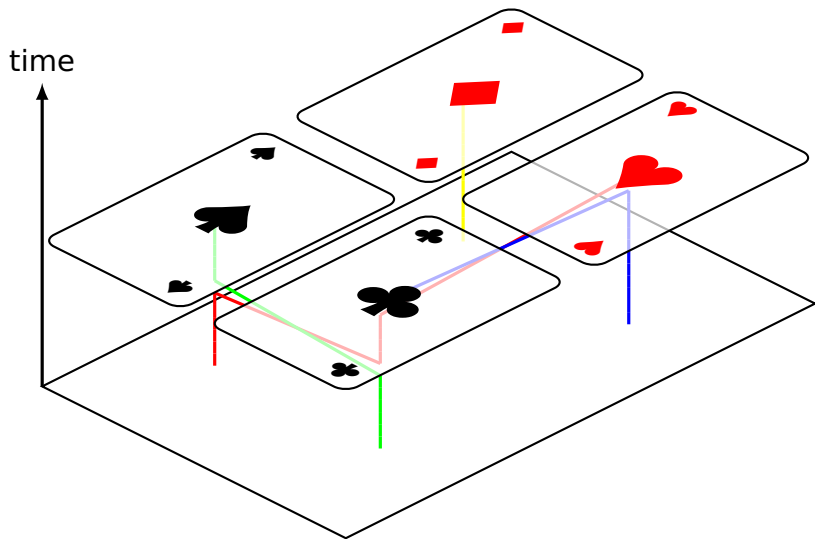
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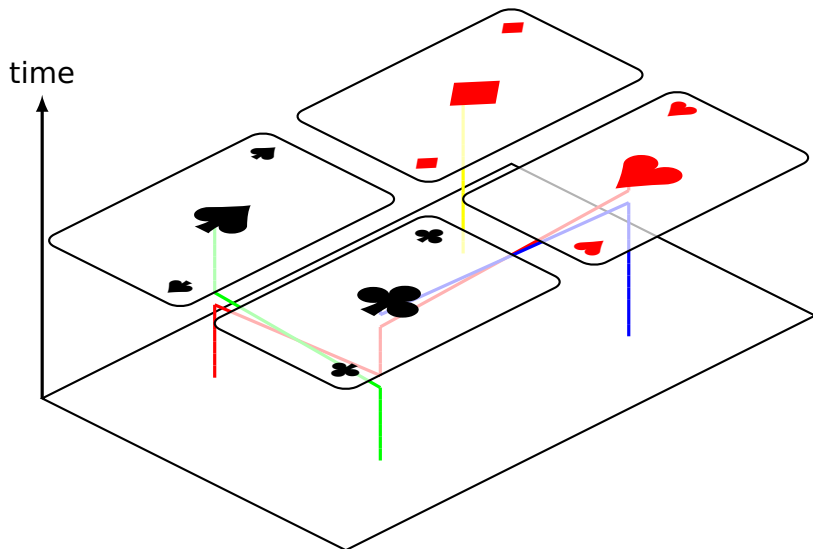
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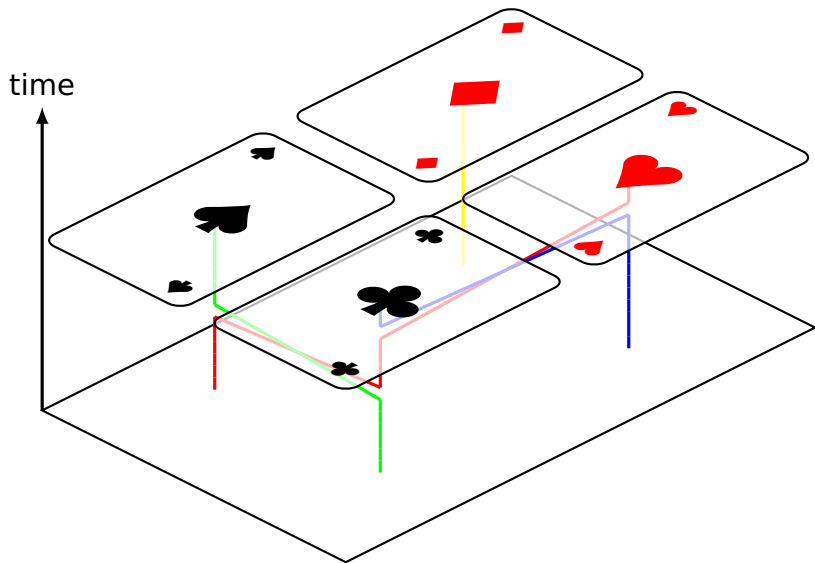
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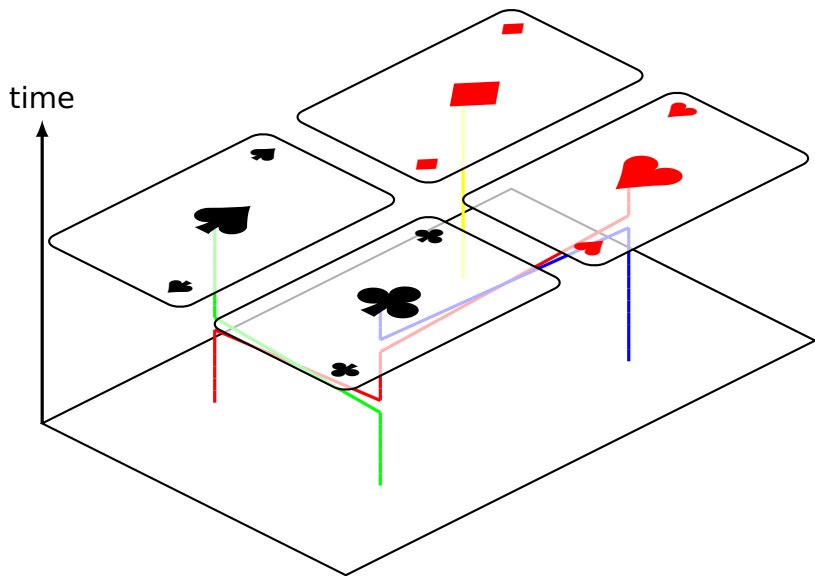
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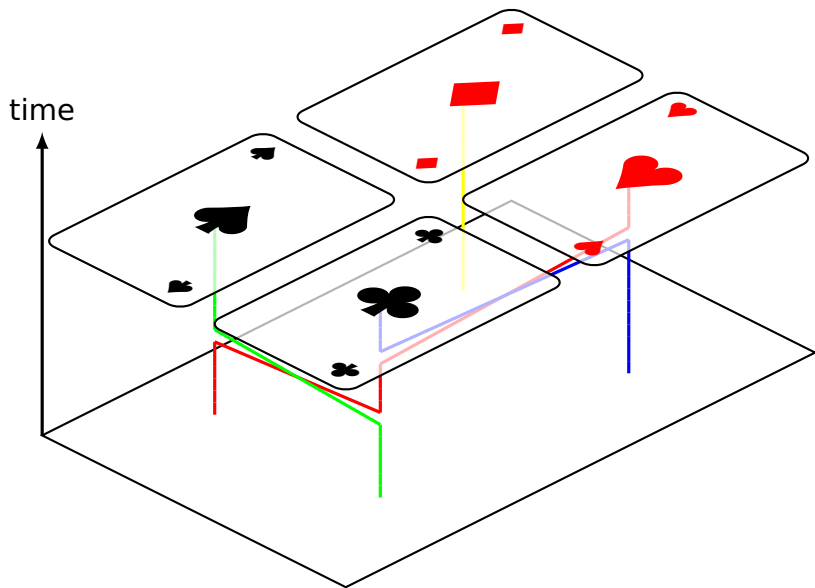
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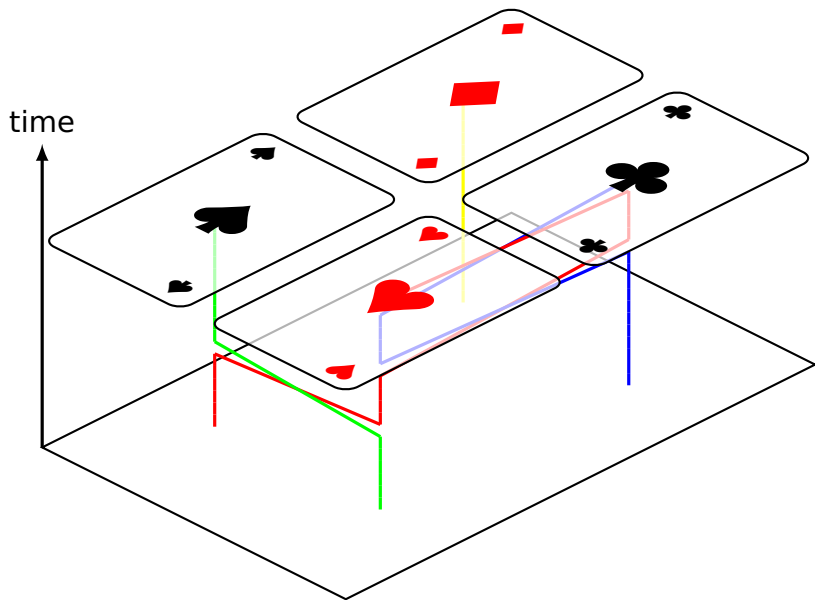
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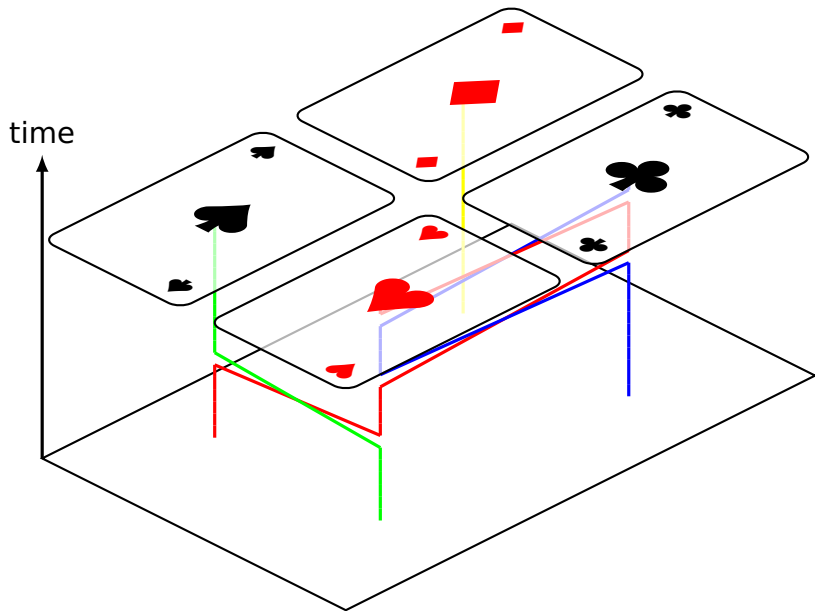
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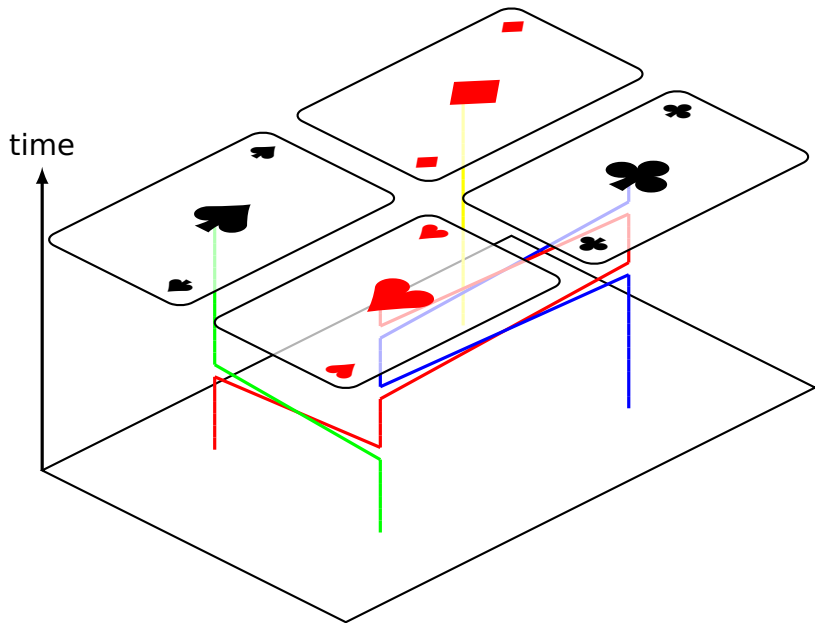
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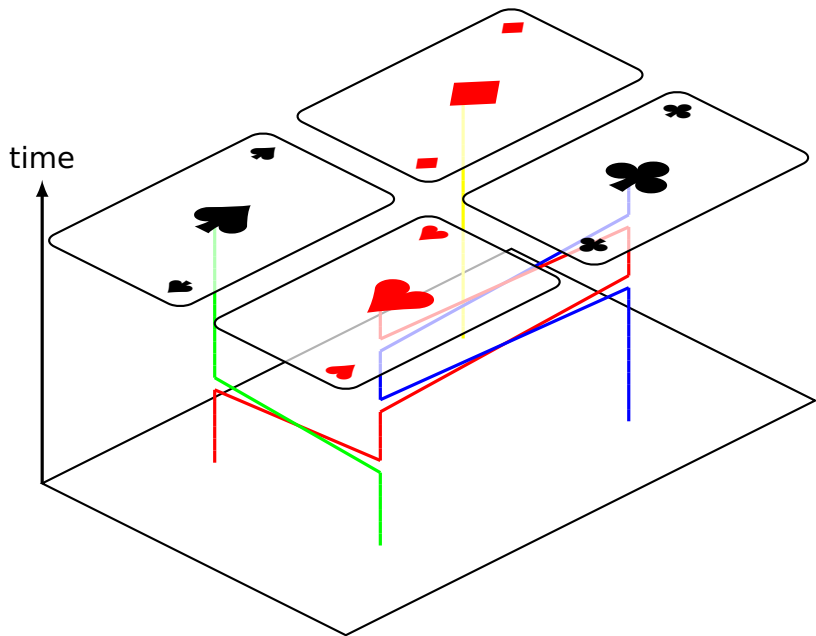
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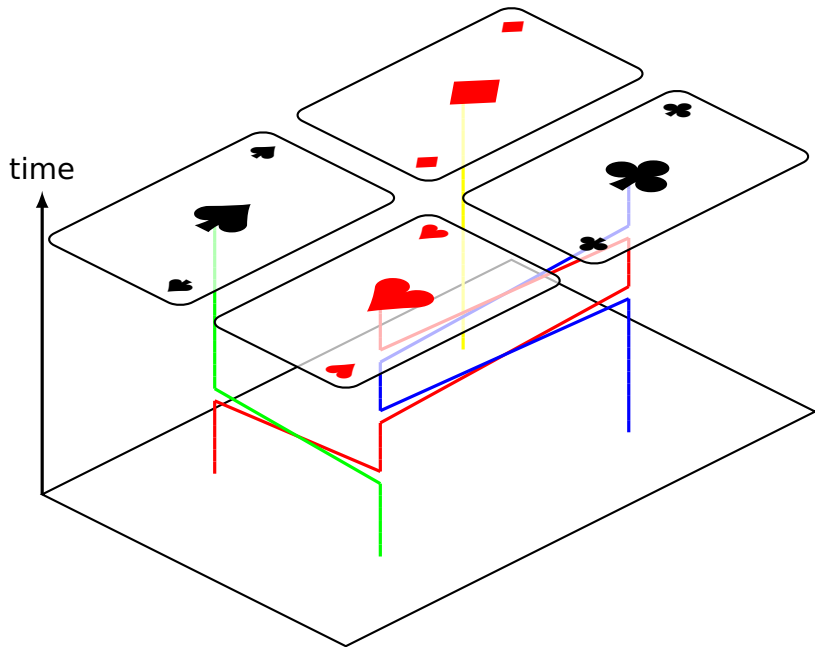
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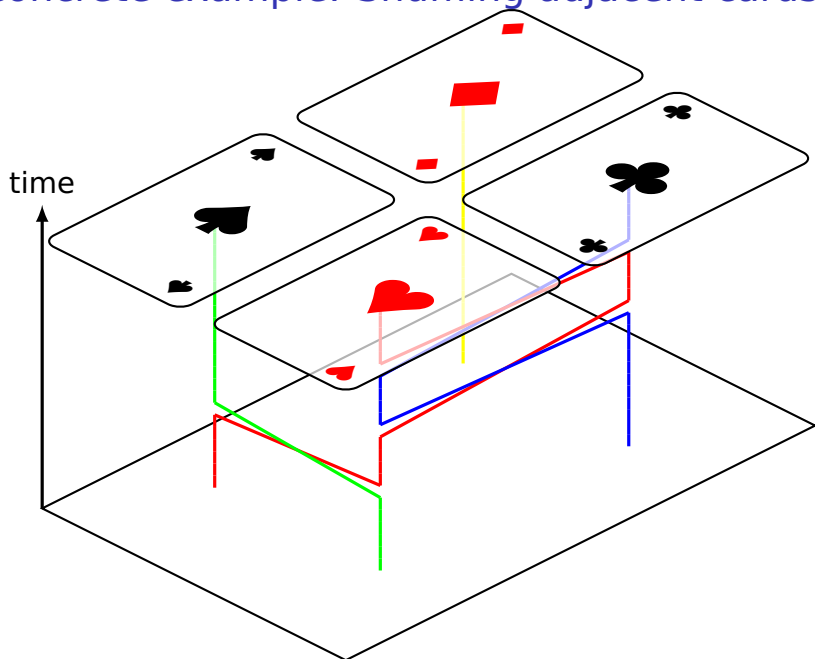
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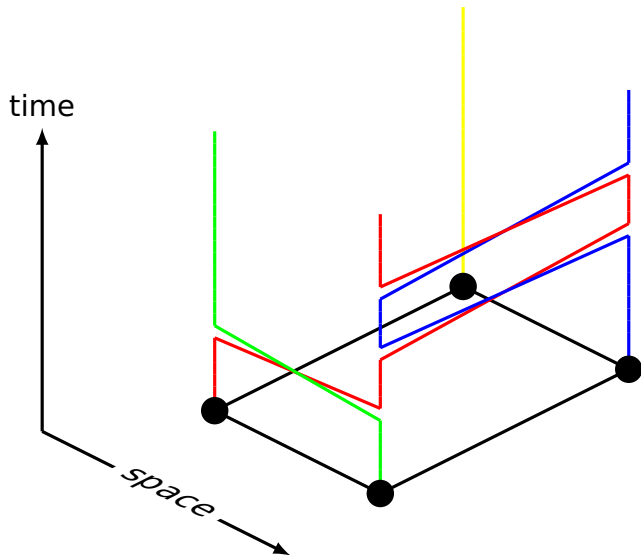
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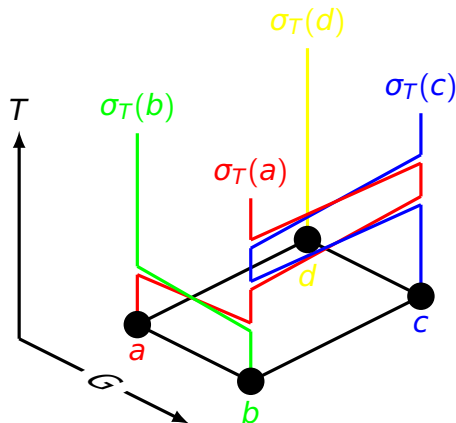


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Definition of the random interchange model

Fix some $T \in (0, \infty)$ and a locally finite graph $G = (V, E)$;
on each edge $e \in E$ place $N_e \sim \text{Poisson}(T)$ crosses at
iid uniform random times $\{t_1, t_2, \dots, t_{N_e}\} \subseteq [0, T)$
 $\Rightarrow$ get random permutation σ_T :

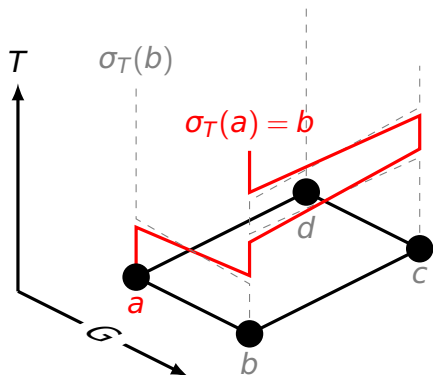


σ_T = product of transpositions supported on edges of G !

What's the point?

This representation lets us relate random permutations to

- ▶ **random walk models** $\Rightarrow$ analyse cycle structure of σ_T
- ▶ percolation $\Rightarrow$ upper bound on large cycles

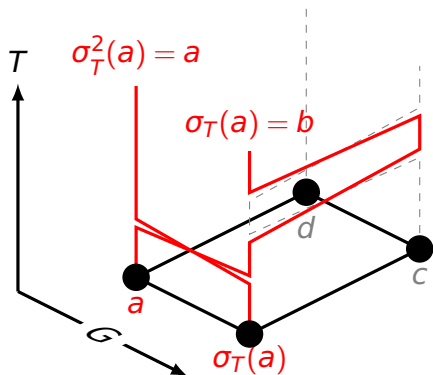


- ▶ Fix any vertex, say a .
- ▶ Cycle with a is $(a, \sigma_T(a), \sigma_T^2(a), \dots)$.

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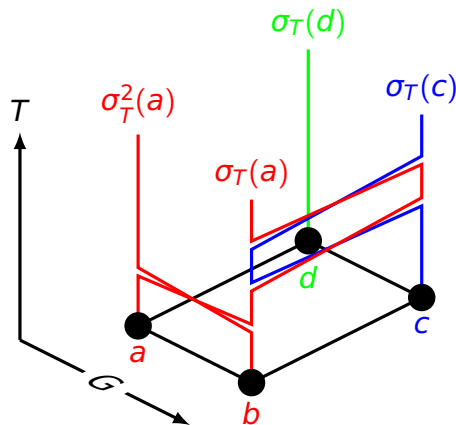


- ▶ Fix any vertex, say a .
- ▶ Cycle with a is $(a, \sigma_T(a), \sigma_T^2(a), \dots)$.
- ▶ Strongly dependent random walks on G !
 $\Rightarrow$ It's **really** hard.

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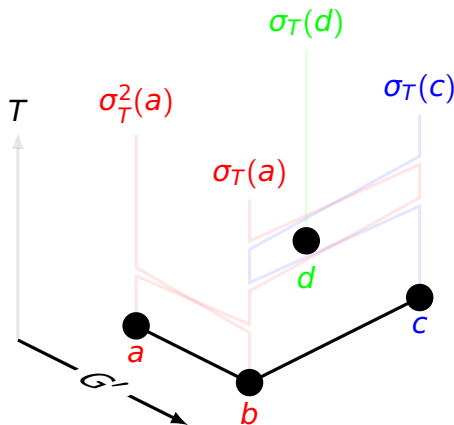


- ▶ Fix any two vertices, say a, d .
- ▶ Are they in the same cycle?

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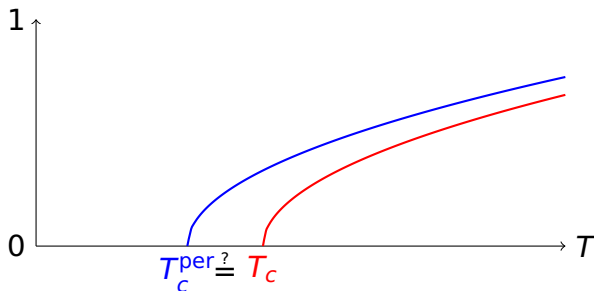
- ▶ Fix any two vertices, say a, d .
- ▶ Are they in the same cycle?
- ▶ Construct G' by keeping only edges with ≥ 1 crosses.
- ▶ **Necessary** condition: There is a path in G' from a to d .

Obtaining a percolation bound on large cycles

Let $Cyc_a := \{a, \sigma_T(a), \dots\}$ and $Clu_a := \{b : a \leftrightarrow_{G'} b\}$.

Then $Cyc_a \subseteq Clu_a$ and hence

$$\mathbb{P}(|Cyc_a| = \infty) \leq \mathbb{P}(|Clu_a| = \infty).$$



G' was constructed by keeping each edge e of G independently with probability $\mathbb{P}(\exists \text{cross on } e) = 1 - e^{-T}$
 $\Rightarrow$ percolation with parameter $p = 1 - e^{-T}$!
In particular there are no infinite cycles for small T .

Why care about whether $T_c = T_c^{\text{per}}$?

- Recall:**
- ▶ T_c corresponds to phase transition.
 - ▶ T_c^{per} is easier to calculate (percolation).

- “Conj”:**
- ▶ $T_c = T_c^{\text{per}}$ on graphs of diverging vertex degree.
 - ▶ $T_c > T_c^{\text{per}}$ on graphs of bounded vertex degree.

- Proved:**
- ▶ $T_c = T_c^{\text{per}}$ on complete graph, hypercube, Hamming graph.
 - ▶ $T_c = T_c^{\text{per}} \pm 2d^{-2}$ as $d \rightarrow \infty$ on d -regular trees.

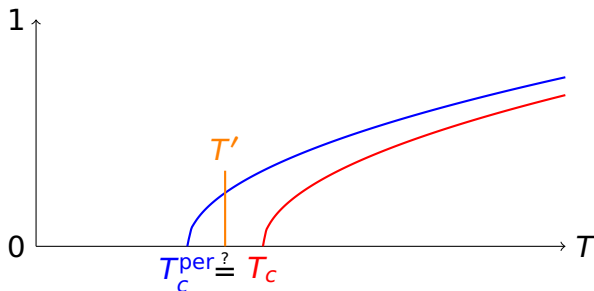
A strict inequality for bounded degree graphs

Theorem

For all countably infinite, connected graphs G of uniformly bounded degree with $p_c(G, \text{bond}) < 1$ we have $T_c > T_c^{\text{per}}$.

Recall:

$$\mathbb{P}(|\text{Cyc}_a| = \infty) \leq \mathbb{P}(|\text{Clu}_a| = \infty)$$



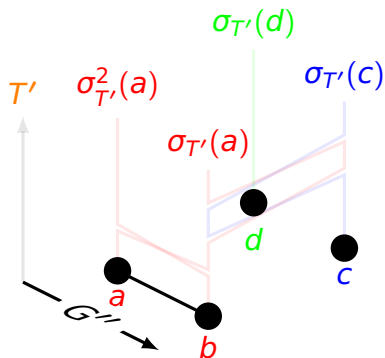
Thm $\Leftrightarrow \exists T' : G'$ percolates, but $\sigma_{T'}$ has no infinite cycles.

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Recall:



Idea: Construct G'' by throwing away edges of G' with "double crosses".

Q: $\exists T' : G'$ percolates, but G'' doesn't? (sufficient)

$\Rightarrow$ Bound from below:
 $(\mathbb{1}\{e \text{ is double cross}\})_{e \in E}$

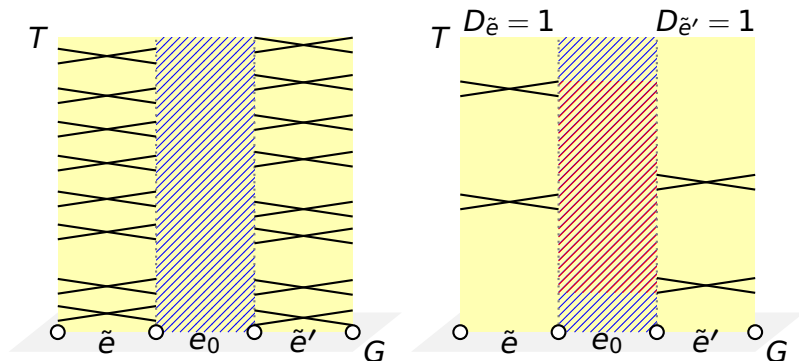
Bounding $(D_e)_{e \in E} := (\mathbb{1}\{e \text{ is double cross}\})_{e \in E}$

Need to show the finite energy property:

$$\mathbb{P}(D_{e_0} = 1 | D_{e_1} = \epsilon_1, \dots, D_{e_m} = \epsilon_m) \geq \delta > 0,$$

for all $m \in \mathbb{N}$, $e_0 \notin \{e_1, \dots, e_m\}$, $\epsilon_i \in \{0, 1\}$.

Potential problems:



Conclusion

- ▶ Introduced the interchange process, an instance of a loop soup.
- ▶ Gave bound on critical parameter for existence of infinite loops in terms of percolation.
- ▶ Original contribution: Showed that this bound is not sharp on graphs of bounded degree, e.g. $\mathbb{Z}^d$.

Thank you for your attention!