

Loop Soups

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Central object

“Def:” A loop soup is a probability measure on the space of multisets of closed trajectories in a space Λ .

Setup of interest:

physical system



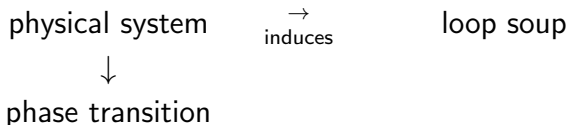
phase transition

We will focus on some finite $\Lambda := \{1, \dots, n\}$.

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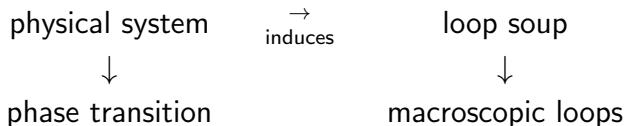


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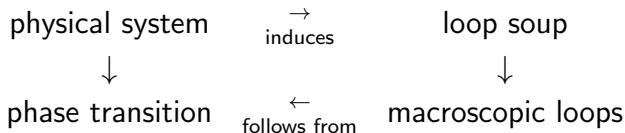


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Existing literature/motivation

Type 1: Feynman '53: Feynman's representation of the interacting Bose gas.

- ▶ Kikuchi '54: Showed phase transition (neglecting interactions).
- ▶ Sütő '93: $\exists$ macroscopic loops $\Rightarrow$ BEC (for ideal gas).

Type 2: BFS '82: Symanzik-BFS loop representation of $O(N)$ models (correlation inequalities, estimates on critical temperature).

- ▶ Fernández-Fröhlich-Sokal '92: “Triviality” of the behaviour of correlation functions.

Type 3: Tóth '93: loop representation of spin $\frac{1}{2}$ quantum Heisenberg ferromagnet (lower bound on thermodynamic pressure).

- ▶ Aizenman-Nachtergaele '94: Same for antiferromagnet.
- ▶ Ueltschi '13: Generalisation to include spin $\frac{1}{2}$ quantum XY model.

How do physical systems induce loop soups?

Some quantity Q that encodes physical properties can be written in terms of loops.

- ▶ partition function Z (Type 1, 2, 3)
- ▶ correlation function $\langle S_x^i S_y^i \rangle$ (Type 2, 3)
- ▶ grand partition function Ξ (Type 3)

Usually this has to be done on a case by case basis, but there are some recurring themes that will be illustrated by the following examples.

Ex 1: Feynman's representation of interacting Bose gas

Consider the partition function $Z := \text{tr } P_{\text{sym}} e^{-\beta H_{\Lambda,n}}$ for

$$H_{\Lambda,n} := -\frac{1}{2} \sum_{i=1}^n \Delta_i + \sum_{i < j}^n U(x_i - x_j).$$

- ▶ Find integral kernel A of $e^{-\beta H_{\Lambda,n}}$ (Trotter product formula).
- ▶ Note that the integral kernel A_{sym} of $P_{\text{sym}} e^{-\beta H_{\Lambda,n}}$ is given by $A_{\text{sym}}(x, y) = \frac{1}{n!} \sum_{\sigma} A(\sigma(x), y)$.
- ▶ Now since “ $\text{tr } B = \int B(x, x) dx$ ” we get

$$Z = \frac{1}{n!} \sum_{\sigma} \int_{\Lambda} A(\sigma(x), x) dx.$$

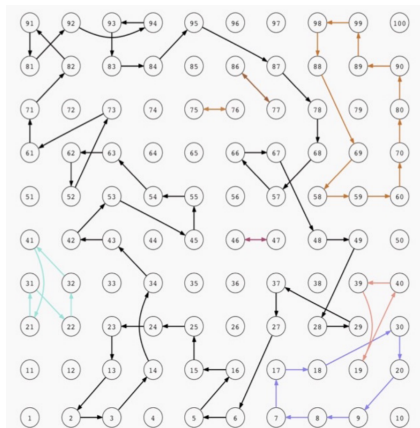
- ▶ So we put a weight $\frac{1}{n!} \int_{\Lambda} A(\sigma(x), x) dx \geq 0$ on each σ .
- ▶ This induces a p.m. on space of permutations.

Takeaway from Ex 1

Look out for

$$Q = \sum_{\sigma \text{ a permutation}} f(\sigma),$$

with $f(\sigma) \geq 0$ naturally induces a p.m. on permutations $\Rightarrow$
induces a p.m. on set of disjoint cycles.



Further remarks

- ▶ To get this representation the trace over the symmetric subspace was essential $\rightarrow$ look out for **bosons**!
Didn't use any properties of $U \rightarrow$ don't expect too many insights.
- ▶ One way of choosing U to get a tractable model: $U \equiv 0$; assume furthermore that particles are evenly spaced on a lattice Λ , then we get
$$Z = \sum_{\sigma \in S_N} \exp \left(\frac{1}{\beta} \sum_{x \in \Lambda} \|x - \sigma(x)\|^2 \right).$$
- ▶ Another way of choosing U : hard-core interactions.

Ex 2: Tóth's model

Look out for hard-core interactions.

Start from Feynman-Kac formula (in discrete setting; N points in $\Lambda \subseteq \mathbb{Z}^d$):

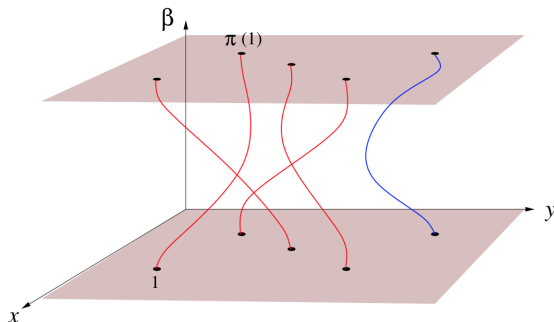


Figure: Feynman-Kac in continuous setting. Graphical illustration of

$$Z = \frac{1}{N!} \sum_{\pi \in S_N} \sum_{x_1, \dots, x_N \in \Lambda} \mathbb{E} \left[e^{-\int_0^\beta \sum_{i < j} U(\omega_i(s) - \omega_j(s)) ds} \mathbf{1}_{\{\omega_i(\beta) = x_{\pi(i)}\}} \middle| \omega_i(0) = x_i \right]$$

Ex 2: Tóth's model $\leftrightarrow$ RSP (sketch)

- ▶ Because of hard-core interactions, points are disjoint at all times s , hence:

$$\{\omega_i(s)\}_{i=1}^N = \sigma_s(A),$$

for some permutation σ_s on Λ and $A := \{\omega_i(0)\}_{i=1}^N$.

- ▶ The evolution of σ_s is then given by putting an exponential clock on each pair of nearest neighbours $\{x, y\}$ and adding the transposition (x, y) to σ_s if the clock rings at time s .
- ▶ With suitably chosen nearest-neighbour interactions get

$$Z = \mathbb{P}(\sigma_\beta(A) = A),$$

$$\Xi_{\beta, \mu} = \mathbb{E}_\sigma \prod_{\ell \geq 1} \left(1 + e^{\beta \mu \ell}\right)^{|\{\text{cycles of length } \ell \text{ in } \sigma_\beta\}|}.$$

Ex 3: Symanzik-BFS loop representation

- ▶ Look out for A^{-1} or $\det A^{-1}$ with some diagonally dominant A .
- ▶ Recall that Gaussian integrals hold for “fairly general” A :

$$\int \phi_x \phi_y e^{-\frac{1}{2} \langle \phi, A \phi \rangle} d\phi = \sqrt{\frac{(2\pi)^n}{\det A}} (A^{-1})_{xy}$$

For proofs assume $A = 1 - P$ with some transition matrix P :

$$\begin{aligned} (1 - P)_{xy}^{-1} &= \sum_{k=1}^{\infty} P_{xy}^k & \det(1 - P)^{-1} &= \exp \{ -\text{tr} \ln(1 - P) \} \\ &= \sum_{\omega: x \rightarrow y} \mathbb{P}(\omega) & &= \exp \left\{ - \sum_{k=1}^{\infty} \frac{1}{k} \text{tr} P^k \right\} \\ & & &= \exp \left\{ - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{x=1}^n \sum_{\substack{\omega: x \rightarrow x \\ |\omega|=k}} \mathbb{P}(\omega) \right\} \\ & & &= \exp \left\{ - \sum_{\tilde{\omega}} \mathbb{P}(\tilde{\omega}) \right\} \end{aligned}$$

Bosons are nice because of Gaussian integrals (Wick's theorem!)

for partition and correlation functions.

See survey on arXiv:[1703.09503](#) for citations.

Thank you for your attention!